An Introduction to Finite Element Analysis Using MATLAB

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Motivation

FEM software → Industrial purpose

Programming FEA → Academic purpose

There is not any limitation

Deep understanding of Finite Element Method

Commercial FEM software is garbage in garbage out

Course Outline

1- Introduction to MATLAB

- MATLAB
- Command vs. Function Syntax
- Data types
- Common Functions and Commands
- M-file vs. Mlx-file
- Simulation Strategy

2- Introduction to FEA

- Basic Concepts
- Applications
- Analysis Procedure

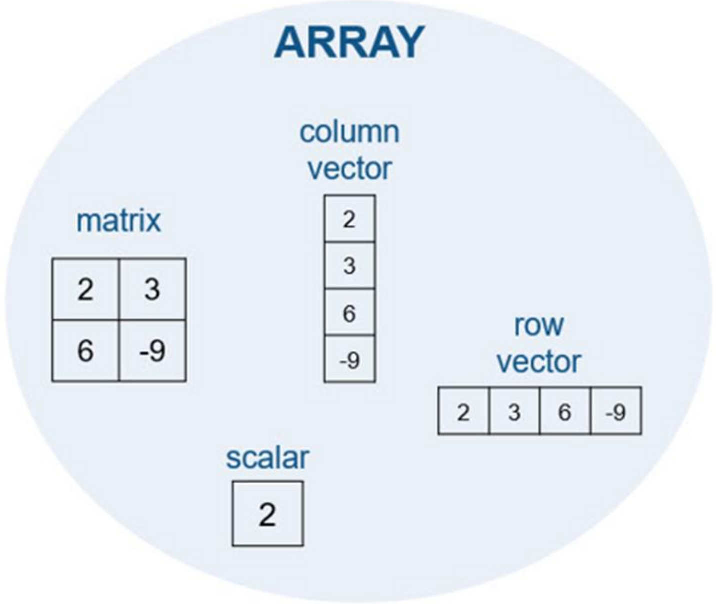
3- Problems

- 2D Truss Problem
- 2D Beam Problem
- 3D Truss Problem
- 2D Frames (2D Column Beam)
- 3D Frames (3D Column Beam)
- Membrane Problem
- Plane Stress Problems
- Axisymmetric Problem
- 2D Transient Heat Transfer Problem
- Thin Plate Problem
- Thick Plate Problem

Introduction to MATLAB: MATLAB

MATLAB is an abbreviation for "**MATrix LABoratory**."

MATLAB is a programming platform designed specifically for **engineers** and **scientists**. The heart of MATLAB is the MATLAB language, a **matrix-based language** allowing the most natural expression of computational mathematics. While other programming languages mostly work with numbers one at a time, MATLAB is designed to operate primarily on whole matrices and arrays.



Introduction to MATLAB: MATLAB Reference

How to write code

- MATLAB Documentation
- doc + function/command
- help + function/command

Introduction to MATLAB: Command vs. Function Syntax

In MATLAB, these statements are equivalent:

Command syntax: load Workspace.mat
Function syntax: load(' Workspace.mat’)

This equivalence is sometimes referred to as command-function **duality**.

All functions support this standard **function syntax**: [output1, ..., outputM] = functionName(input1, ..., inputN)

If you do not require any outputs from the function, and all of the inputs are character vectors (that is, text enclosed in single quotation marks), you can use this simpler **command syntax**: functionName input1 ... inputN

With command syntax, **you separate inputs with spaces rather than commas**, and do not enclose input arguments in parentheses. Command syntax always passes inputs as **character vectors**.

To use strings as inputs, use the function syntax.
If a character vector contains a space, use the function syntax.

When a function input is a **variable**, you must use function syntax to pass the value to the function. Command syntax always passes inputs as **character** vectors and **cannot pass variable values**.

Introduction to MATLAB: Data types

By default, MATLAB stores all numeric variables as **double-precision floating-point** values.
Additional data types store **text**, **integer** or **single-precision** values, or a combination of related data in a single variable

Numeric Types: Integer and floating-point data

Characters and Strings: Text in character arrays (' ') and string arrays (" ")

Dates and Time: Arrays of date and time values that can be displayed in different formats

Categorical Arrays: Arrays of qualitative data with values from a finite set of discrete, nonnumeric data

Tables: Arrays in tabular form whose named columns can have different types

Timetables: Time-stamped data in tabular form

Structures: Arrays with named fields that can contain data of varying types and sizes

Cell Arrays: Arrays that can contain data of varying types and sizes

Function Handles: Variables that allow you to invoke a function indirectly

Map Containers: Objects with keys that index to values, where keys need not be integers

Time Series: Data vectors sampled over time

Data Type Identification: Determining data type of a variable

Data Type Conversion: Converting between numeric arrays, character arrays, cell arrays, structures, or tables

Introduction to MATLAB: Common Functions and Commands

Most Common MATLAB code

ans	Most recent answer
clc	Clear Command Window
clear	Clear Workspace
global	Declare variables as global
plot	2-D line plot
format	Set Command Window output display format
iskeyword	Determine whether input is MATLAB keyword
fpritf/sprintf	Write data to text file/Format data into string or character vector
zeros	Create array of all zeros
ones	Create array of all ones
eye/diag	Identity matrix/Creates or extract diagonals
fopen	Open file, or obtain information about open files
fcolse	Close one or all open files
patch	Plot one or more filled polygonal regions

Introduction to MATLAB: Common Functions and Commands

1-Matrices can be created in MATLAB by the command

```
>> A=[1 2 3;4 5 6;7 8 9]
A =
     1     2     3
     4     5     6
     7     8     9
```

Note the semi-colon at the end of each matrix line.

2-Operating with matrices

3-Statements: are operators, functions and variables, always producing a matrix which can be used later.

4-Matrix functions

eye	Identity matrix
zeros	A matrix of zeros
ones	A matrix of ones
diag	Creates or extract diagonals
rand	Random matrix

5-Conditionals, if and switch

```
x=-1
if x==0
    disp('Bad input!')
elseif max(x) > 0
    y = x+1;
else
    y = x^2;
end
```

```
switch units
    case 'length'
        disp('meters')
    case 'volume'
        disp('cubic meters')
    case 'time'
        disp('hours')
    otherwise
        disp('not interested')
end
```

6-Loops: for and while

7- Relations

8-Submatrix

9-Logical indexing

Introduction to MATLAB: M-file vs. Mlx-file

M-file:

Plain Code Scripts and Functions

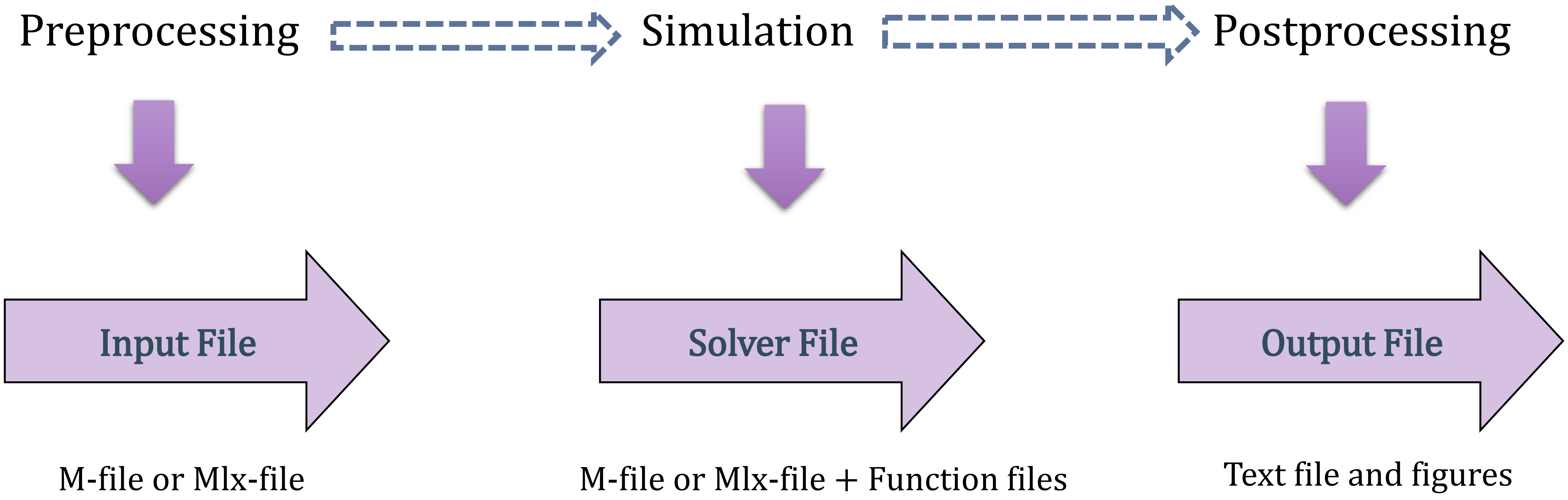
In new Versions: Functions could be saved as separate m-files (function) as well as in the end off main script

Mlx-file:

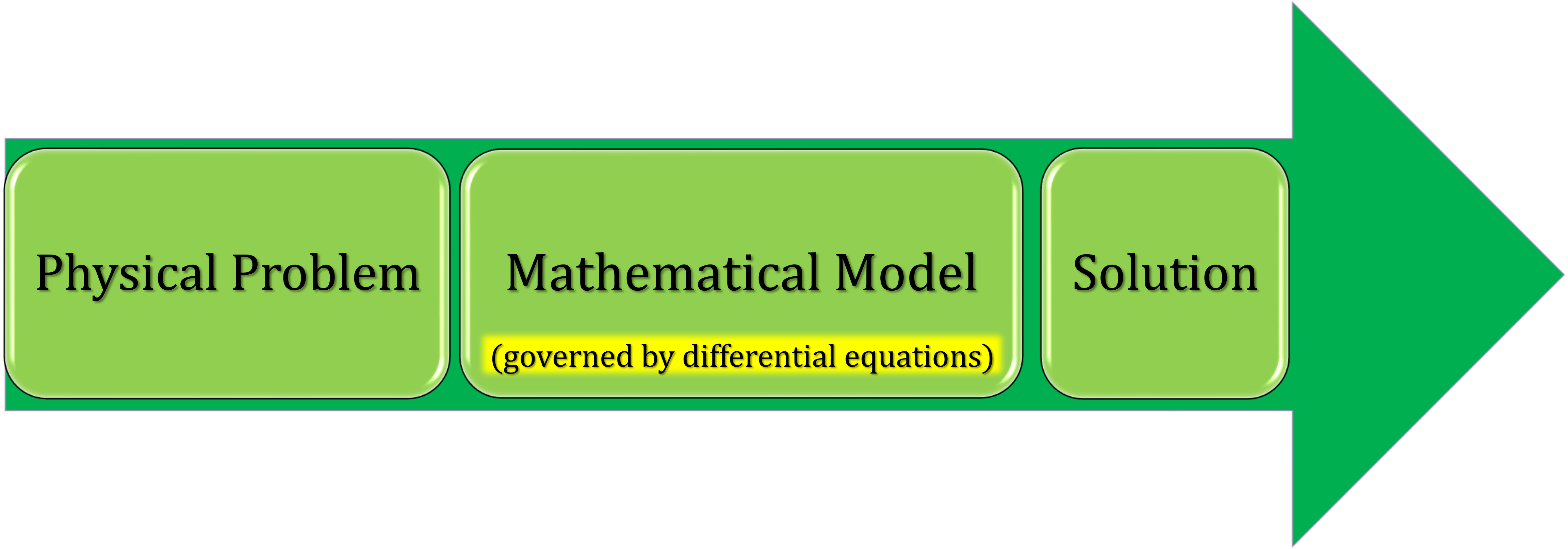
MATLAB live scripts and live functions are interactive documents that combine MATLAB code with formatted text, equations, and images in a single environment called the Live Editor. In addition, live scripts store and display output alongside the code that creates it.

Functions could be saved as separate mlx-files (function) as well as in the end off main script

Introduction to MATLAB: Simulation Strategy



Introduction to FEA: Basic Concepts



Introduction to FEA: Basic Concepts

Methods of Analysis

Analytical Methods

- ODE
- PDE → Separation of variables

Semi-analytical (Approximate) Methods

- Lumped-parameter Methods
- **Series Discretization Methods**

Numerical Methods

- Numerical Integration
- Finite Volume Method
- **Finite Element Method**
- Finite Difference Method
- Boundary Element Method

The existing mathematical tools will not be sufficient to find the exact solution (and sometimes, even an approximate solution) of most of the practical problems.

Introduction to FEA: Basic Concepts

Analytical Methods

Semi-analytical (Approximate) Methods

Series Discretization Methods = Assumed Solution +

- Variational Approach → Weak Form
- Weighted Residual Approach → Strang Form

Must be satisfied
Essential (geometry)
Boundary conditions

Must be satisfied
Essential (geometry)
as well as
Natural (force)
Boundary conditions

Numerical Methods

Finite Element Method

- Weak Form
- Strang Form

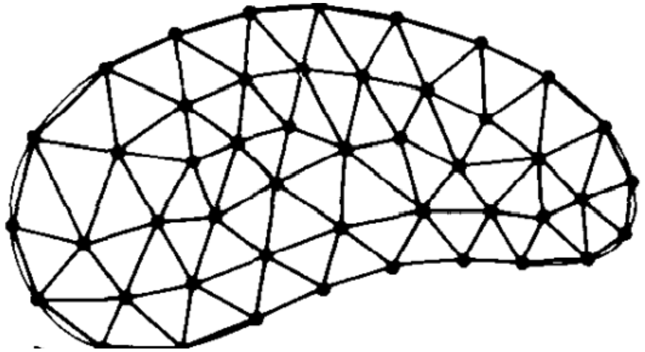
Introduction to FEA: Basic Concepts

What is Finite Element Analysis ?

The Finite Element Analysis (FEA) is the simulation of any given physical phenomenon using the **numerical technique** called Finite Element Method (FEM).

The basic idea behind the finite element method is **to divide** the structure, body, or region being analyzed into a large number of finite elements, or simply elements.

The solution region is considered to be built of many small, interconnected subregions called elements.



Space Discretization

↓

FEM **subdivides** a large system into smaller, simpler parts that are called **finite elements**

↓

construction of a **mesh** of the object

Introduction to FEA: Applications

Structural Analysis

Thermal Analysis

Fluid Structure Analysis

Electromagnetic Analysis

Multiphysics Analysis

Optimization Analysis

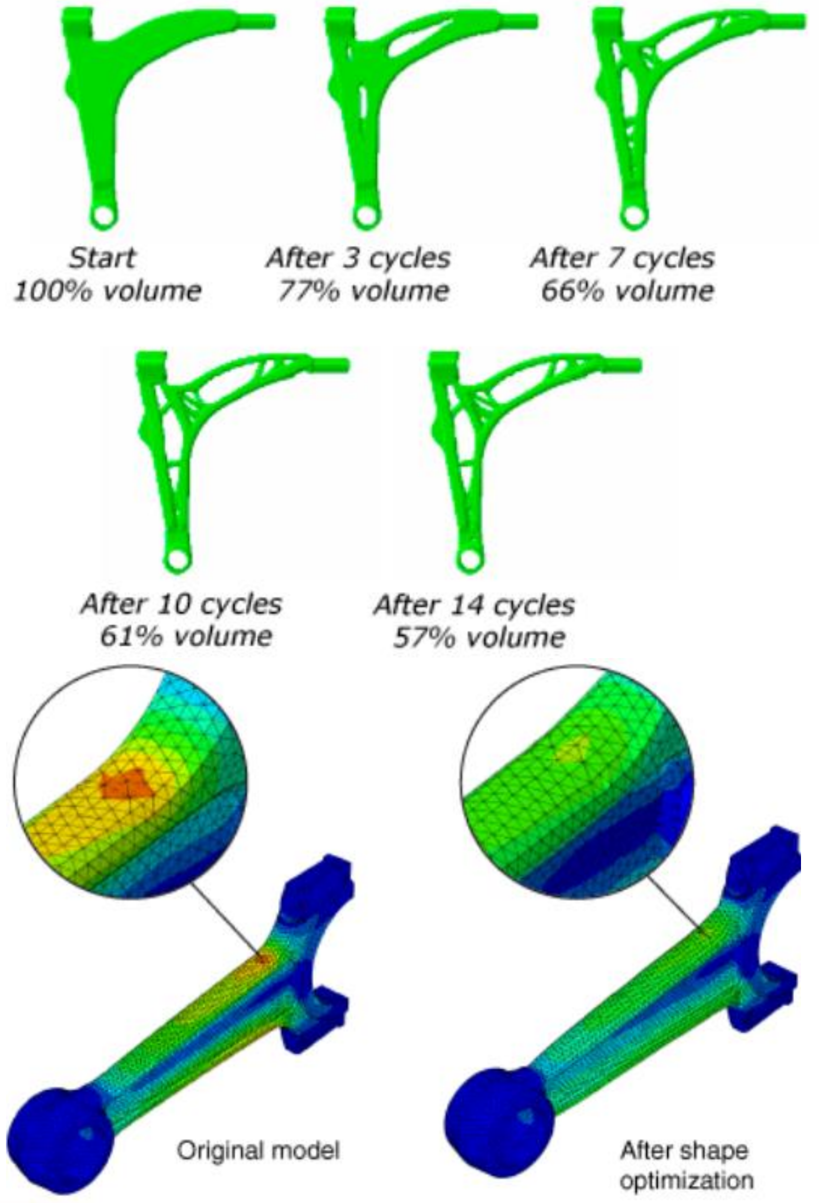
Introduction to FEA: Applications

Structural Analysis

- time-independent
  - Static → Displacement/Stress Analysis → Sandwich panel; Composite Material
  - Eigenvalue problem
    - Buckling and post-buckling analysis
    - Frequency and mode shape analysis
- time-dependent
  - Quasi-static
    - Time-depend material response → Creep and Viscoelasticity
    - Low-velocity forming
    - Unstable problems → local instabilities (e.g. surface wrinkling) and local buckling
  - Dynamic
    - Explicit dynamic analysis → Impact of Composite Material
    - Implicit dynamic analysis
    - Response spectrum analysis

Introduction to FEA: Applications

- Analysis
  - Multiphysics
    - Thermal structural analysis
    - Fluid structural analysis
  - Optimization
    - Topology optimization
    - Shape optimization



Introduction to FEA: Analysis Procedures

Procedures

1-Discretization

2-Interpolation (Shape Function)

3-Derivation of characteristic matrices (element stiffness matrices and load vectors)

4-Assembly

5-Applying Boundary Conditions

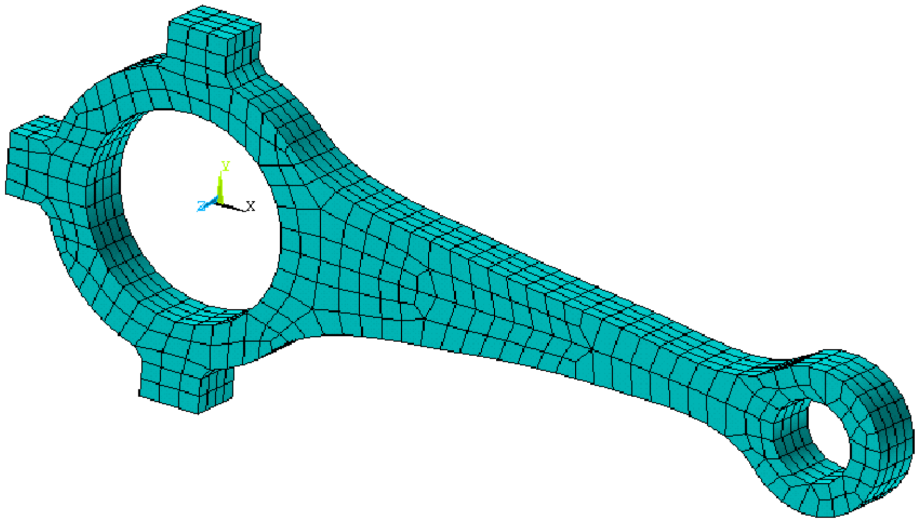
6-Solving unknown

Introduction to FEA: Analysis Procedures

1- Discretization

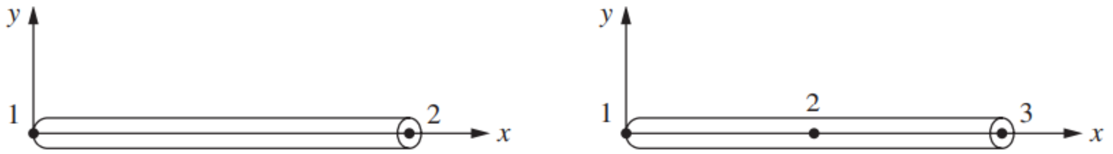
The **first step** in the finite element method involves dividing the body into an equivalent system of **finite elements** with associated **nodes** and choosing the most appropriate **element type** to model most closely the actual physical behavior.

Small elements (and possibly higher-order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur

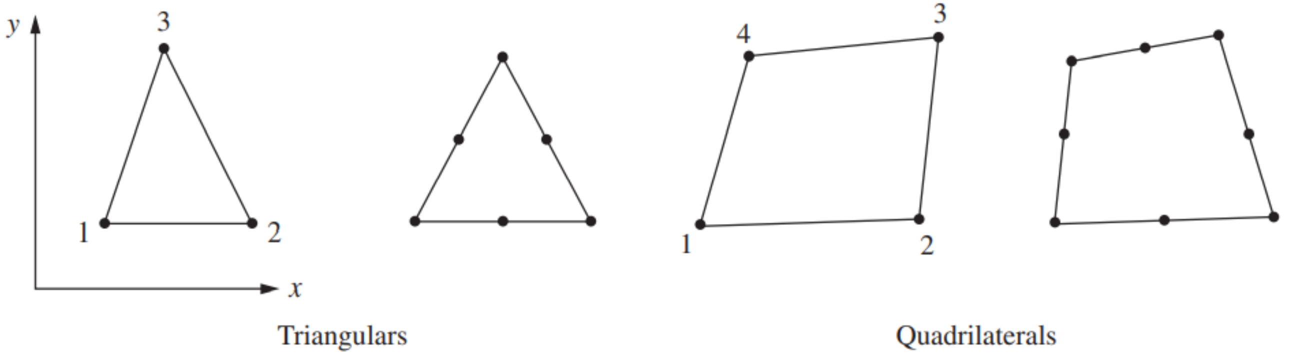


Spatial Discretization (Mesh)

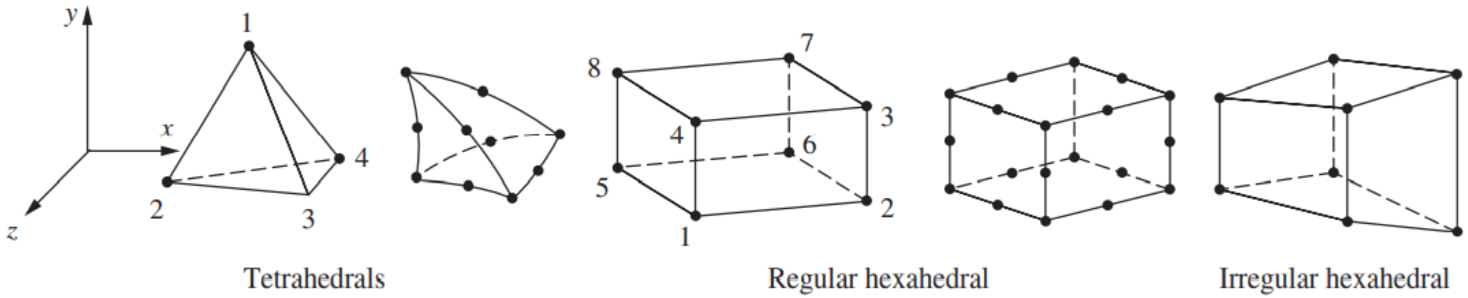
Introduction to FEA: Analysis Procedures



(a) Simple two-noded line element (typically used to represent a bar or beam element) and the higher-order line element



(b) Simple two-dimensional elements with corner nodes (typically used to represent plane stress/strain) and higher-order two-dimensional elements with intermediate nodes along the sides



(c) Simple three-dimensional elements (typically used to represent three-dimensional stress state) and higher-order three-dimensional elements with intermediate nodes along edges

Introduction to FEA: Analysis Procedures

2-Interpolation (Select a Displacement Function)

Since the displacement solution of a **complex structure** under any specified load conditions cannot be predicted exactly, we **assume** some suitable solution within an element to **approximate the unknown solution.** The assumed solution must be **simple** from a computational standpoint, but it should satisfy certain **convergence requirements**. In general, the solution or the interpolation model is taken in the **form of a polynomial**.

Approximate Solution $$u(x,y,z) = \sum_{i=1} a_i N_i(x,y,z) = a_1 N_1(x,y,z) + a_2 N_2(x,y,z) + \cdots$$ satisfy the **Essential** boundary conditions exactly

$$u(x,y,z) = [N(x,y,z)]\{a\}$$

Interpolation
(Geometric Order of Element)

Linear Quadratic

Introduction to FEA: Analysis Procedures

Five aspects of an element characterize its behavior:

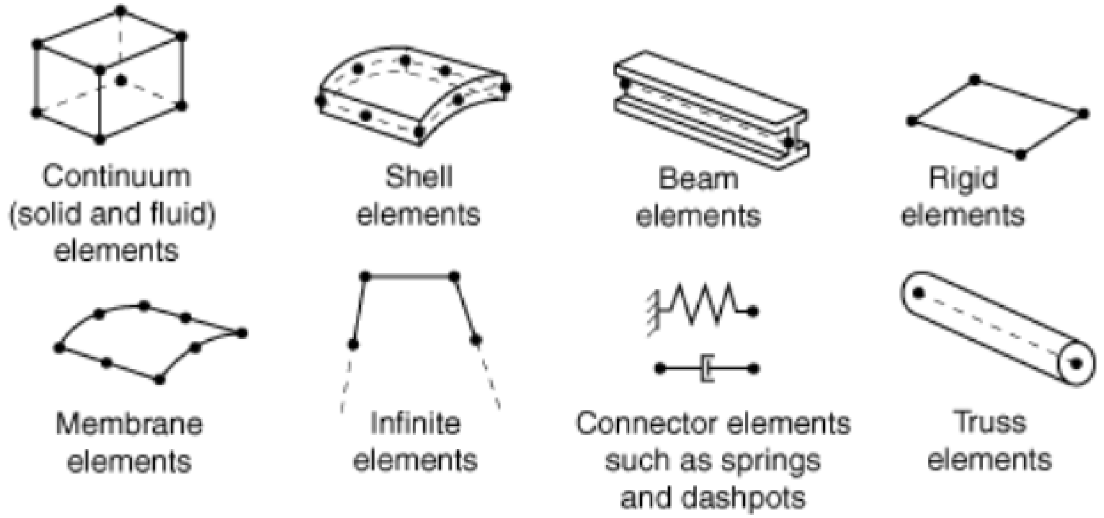
Family

Degrees of freedom Number of nodes

Number of nodes and order of interpolation

Formulation

Integration



Introduction to FEA: Analysis Procedures

Five aspects of an element characterize its behavior:

Family

Degrees of freedom Number of nodes: the translations and, for shell, pipe, and beam elements, the rotations at each node.

Number of nodes and order of interpolation

Formulation

Integration

Introduction to FEA: Analysis Procedures

Five aspects of an element characterize its behavior:

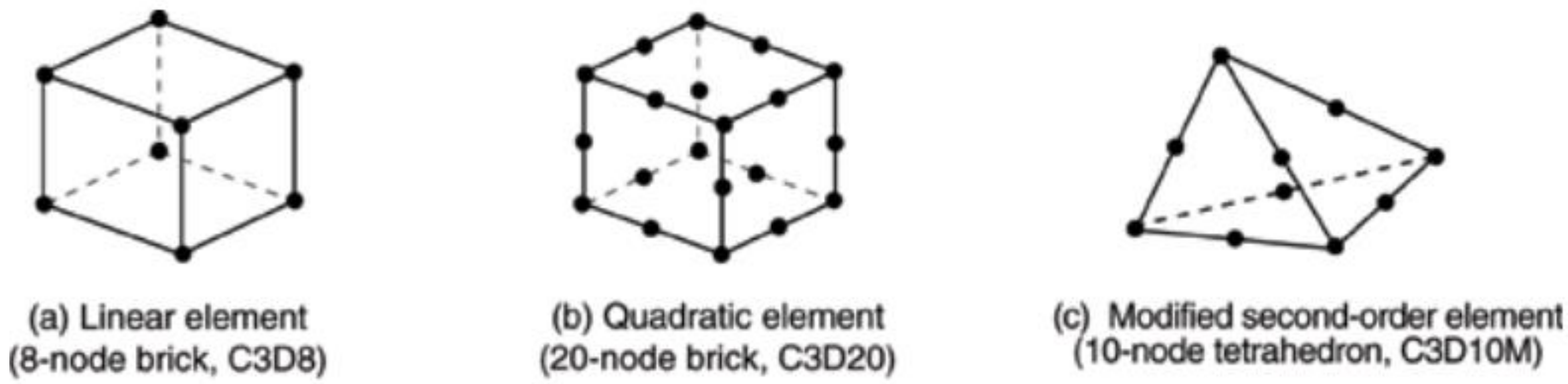
Family

Degrees of freedom Number of nodes

Number of nodes and order of interpolation

Formulation

Integration



Introduction to FEA: Analysis Procedures

Five aspects of an element characterize its behavior:

Family

Degrees of freedom Number of nodes

Number of nodes and order of interpolation

Formulation: mathematical theory used to define the element's behavior (Lagrangian or Eulerian/shell element: 1-general-purpose shell analysis, 2-thin shells, 3-for thick shells.)

Integration

Plane strain	Small-strain shells
Plane stress	Finite-strain shells
Hybrid elements	Thick shells
Incompatible-mode elements	Thin shells

Introduction to FEA: Analysis Procedures

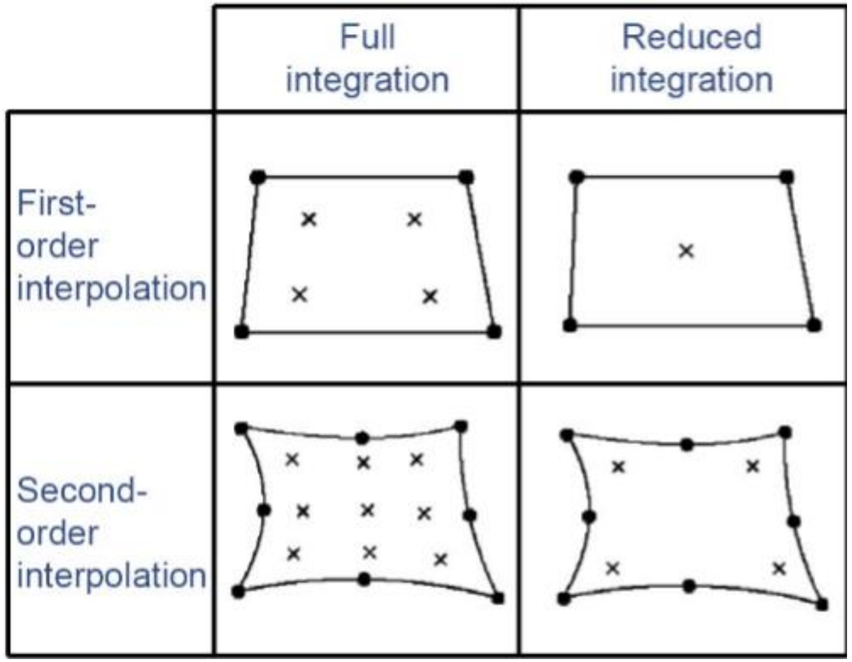
Five aspects of an element characterize its behavior:

Family

Degrees of freedom Number of nodes

Number of nodes and order of interpolation

Formulation



Integration: Using Gaussian quadrature for most elements (full or reduced integration)

Introduction to FEA: Analysis Procedures

3-Derive element stiffness matrices and load vectors

From the assumed displacement model, the stiffness matrix $[K^e]$ and the load vector $\{P^e\}$ of element e are to be derived by using a suitable **variational principle**, a **weighted residual approach** (such as the Galerkin method), or **equilibrium** (direct method) conditions.

FE Formulation

- **Direct Approach:** → Equilibrium
- **Variational Approach:** → Minimizing Functional
- **Weighted Residual Approach:** → Minimizing Error in domain

Introduction to FEA: Analysis Procedures

Direct Approach

According to this method, the stiffness matrix and element equations relating nodal forces to nodal displacements are obtained using force equilibrium conditions for a basic element, along with force/deformation relationships.

Variational Approach

The variational approach is based on the application of variational calculus, which deals with the extremization of functionals in the form of integrals.

$$I = U(u, v, w, \dots) - W_{ext}(u, v, w, \dots) \Longrightarrow I = U(\{a\}) - W_{ext}(\{a\}) \Longrightarrow \delta I = 0 \Longrightarrow \frac{\partial I}{\partial a_i} = 0$$

Weighted Residual Approach

The weighted residual methods allow the finite element method to be applied directly to any differential equation.

$$L(u) + F(x, y, z) = 0 \Longrightarrow R = L(u = [N]\{a\}) + F(x, y, z) \Longrightarrow \int_V w_i R \, dV = 0$$

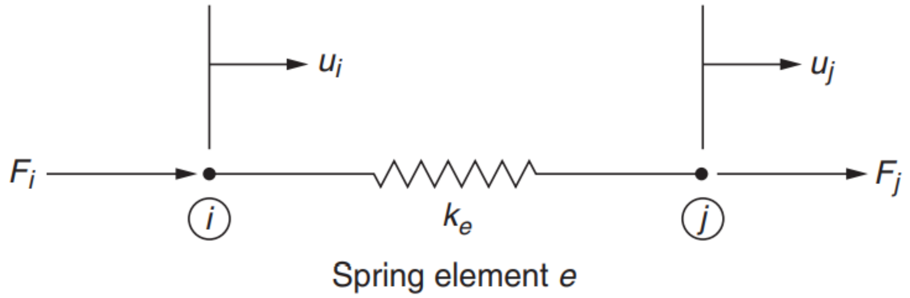
Direct Approach

According to this method, the stiffness matrix and element equations relating nodal forces to nodal displacements are obtained using force equilibrium conditions for a basic element, along with force/deformation relationships.

$$\text{Force} = \text{Spring stiffness} \times \text{Net deformation of the spring}$$

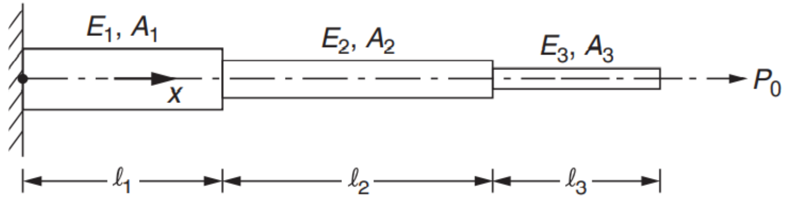
$$F_i = k_e(u_i - u_j)$$
$$F_j = k_e(u_j - u_i)$$

$$\Rightarrow \quad k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$



As an example

$$[K^{(e)}] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} (A_eE_e/l_e) & -(A_eE_e/l_e) \\ -(A_eE_e/l_e) & (A_eE_e/l_e) \end{bmatrix} = \frac{A_eE_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



Variational Approach

$$\delta U = \delta W_{ext} \Longrightarrow \iiint_V \{\boldsymbol{\delta\varepsilon}\}^T\{\boldsymbol{\sigma}\}\, \boldsymbol{dV} = \iiint_V \{\boldsymbol{\delta U}\}^T\{\boldsymbol{F_b}\}\boldsymbol{dV} + \iint_S \{\boldsymbol{\delta U}\}^T\{\boldsymbol{T}\}\boldsymbol{dS} + \sum_{i=1}^{n}\{\boldsymbol{\delta U}\}^T\{\boldsymbol{F_p}\}$$

Stiffness matrix

Self Strain

Stress Vector

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{D}](\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon_0}\}) + \{\boldsymbol{\sigma_0}\}$$

Prestress Vector

Total Strain

Elastic strain energy

Prestress energy

Surface Traction work

$$\iiint_V \{\boldsymbol{\delta\varepsilon}\}^T[\boldsymbol{D}]\{\boldsymbol{\varepsilon}\}\, \boldsymbol{dV} - \iiint_V \{\boldsymbol{\delta\varepsilon}\}^T[\boldsymbol{D}]\{\boldsymbol{\varepsilon_0}\}\boldsymbol{dV} + \iiint_V \{\boldsymbol{\delta\varepsilon}\}^T\{\boldsymbol{\sigma_0}\}\boldsymbol{dV} - \iiint_V \{\boldsymbol{\delta U}\}^T\{\boldsymbol{F_b}\}\boldsymbol{dV} - \iint_S \{\boldsymbol{\delta U}\}^T\, \boldsymbol{T}\, \boldsymbol{dS} - \sum_{i=1}^{n}\{\boldsymbol{\delta U}\}^T\{\boldsymbol{F_p}\} = 0$$

Self strain energy

Body force work

Point Load work

$$\{u\} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = [N(x,y,z)]\{a\}$$

$$\{\varepsilon\} = [L]\{u\} = [L][N(x,y,z)]\{a\} = [B]\{a\}$$

$$\left(\iiint_V \{\boldsymbol{B}\}^T[\boldsymbol{D}]\{\boldsymbol{B}\}\, \boldsymbol{dV}\right)\{\boldsymbol{a}\} = \iiint_V \{\boldsymbol{B}\}^T[\boldsymbol{D}]\{\boldsymbol{\varepsilon_0}\}\boldsymbol{dV} - \iiint_V \{\boldsymbol{B}\}^T\{\boldsymbol{\sigma_0}\}\boldsymbol{dV} + \iiint_V \{\boldsymbol{N}\}^T\{\boldsymbol{F_b}\}\boldsymbol{dV} + \iint_S \{\boldsymbol{N}\}^T\{\boldsymbol{T}\}\boldsymbol{dS} + \sum_{i=1}^{n}\{\boldsymbol{N}\}^T\{\boldsymbol{F_p}\}$$

Weighted Residual Approach

The weighted residual method is a technique that can be used to **obtain approximate solutions** to linear and nonlinear differential equations. If we use this method the finite element equations can be derived directly from the **governing differential equations** of the problem without any need of knowing the functional. We first **consider the solution of equilibrium**, eigenvalue, and propagation problems using the weighted residual method and then derive the finite element equations using the weighted residual approach.

Weighted Residual
- Point Collocation Method
- Subdomain Collocation Method
- Least Squares Method
- **Galerkin Method**

✓ **Galerkin Method**

$$L(\{u\}) + F(x,y,z) = 0 \Longrightarrow R = L(\{u\} = [N]\{a\}) + F(x,y,z) \Longrightarrow \int_V N_i\, R\, dV = 0 \quad i = 1,..,N$$

Introduction to FEA: Analysis Procedures

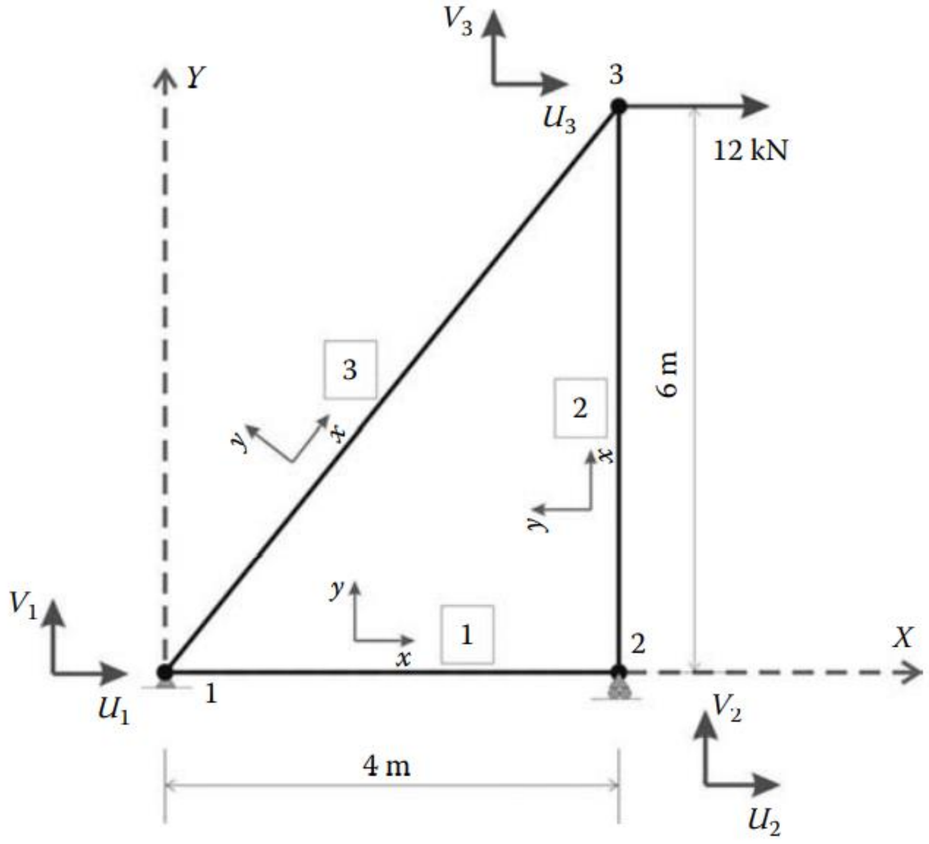
4-Assemble element equations to obtain the overall equilibrium equations

The individual element nodal equilibrium equations are assembled into the global nodal equilibrium equations.

$$\begin{bmatrix} \frac{AE}{L} & 0 & -\frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & \frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix}$$

$$[K_1]_L = \begin{bmatrix} 115000 & 0 & -115000 & 0 \\ 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_2]_L = \begin{bmatrix} 76666.67 & 0 & -76666.67 & 0 \\ 0 & 0 & 0 & 0 \\ -76666.67 & 0 & 76666.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [K_3]_L = \begin{bmatrix} 63791.43 & 0 & -63791.43 & 0 \\ 0 & 0 & 0 & 0 \\ -63791.43 & 0 & 63791.43 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Introduction to FEA: Analysis Procedures

$$[K_1]_L = \begin{bmatrix} 115000 & 0 & -115000 & 0 \\ 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [C_1] = \begin{bmatrix} \cos(0) & -\sin(0) & 0 & 0 \\ \sin(0) & \cos(0) & 0 & 0 \\ 0 & 0 & \cos(0) & -\sin(0) \\ 0 & 0 & \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K_1]_G = \begin{array}{c} \\ U_1/u_1 \\ V_1/v_1 \\ U_2/u_2 \\ V_2/v_2 \end{array} \begin{array}{c} \begin{array}{cccc} U_1/u_1 & V_1/v_1 & U_2/u_2 & V_2/v_2 \end{array} \\ \begin{bmatrix} 115000 & 0 & -115000 & 0 \\ 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[K_2]_L = \begin{bmatrix} 76666.67 & 0 & -76666.67 & 0 \\ 0 & 0 & 0 & 0 \\ -76666.67 & 0 & 76666.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [C_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[K_2]_G = \begin{array}{c} U_2/u_2 \\ V_2/v_2 \\ U_3/u_3 \\ V_3/v_3 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 76666.67 & 0 & -76666.67 \\ 0 & 0 & 0 & 0 \\ 0 & -76666.67 & 0 & 76666.67 \end{bmatrix}$$

$$[K_3]_L = \begin{bmatrix} 63791.43 & 0 & -63791.43 & 0 \\ 0 & 0 & 0 & 0 \\ -63791.43 & 0 & 63791.43 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [C_3] = \begin{bmatrix} 0.554699 & -0.832051 & 0 & 0 \\ 0.832051 & 0.554699 & 0 & 0 \\ 0 & 0 & 0.554699 & -0.832051 \\ 0 & 0 & 0.832051 & 0.554699 \end{bmatrix}$$

$$[K_3]_G = \begin{array}{c} \\ U_1/u_1 \\ V_1/v_1 \\ U_3/u_2 \\ V_3/v_2 \end{array} \begin{array}{c} \begin{array}{cccc} U_1/u_1 & V_1/v_1 & U_3/u_2 & V_3/v_2 \end{array} \\ \begin{bmatrix} 19628 & 29442 & -19628 & -29442 \\ 29442 & 44163 & -29442 & -44163 \\ -19628 & -29442 & 19628 & 29442 \\ -29442 & -44163 & 29442 & 44163 \end{bmatrix} \end{array}$$

Introduction to FEA: Analysis Procedures

$$[K_1]_G = \begin{array}{c} \\ U_1/u_1 \\ V_1/v_1 \\ U_2/u_2 \\ V_2/v_2 \end{array} \begin{array}{cccc} U_1/u_1 & V_1/v_1 & U_2/u_2 & V_2/v_2 \\ 115000 & 0 & -115000 & 0 \\ 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ 115000 & 0 & -115000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$[K_2]_G = \begin{array}{c} U_2/u_2 \\ V_2/v_2 \\ U_3/u_3 \\ V_3/v_3 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 76666.67 & 0 & -76666.67 \\ 0 & 0 & 0 & 0 \\ 0 & -76666.67 & 0 & 76666.67 \end{bmatrix}$$

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 76666.67 & 0 & -76666.67 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -76666.67 & 0 & 76666.67 \end{array}$$

$$[K_3]_G = \begin{array}{c} \\ U_1/u_1 \\ V_1/v_1 \\ U_3/u_2 \\ V_3/v_2 \end{array} \begin{array}{cccc} U_1/u_1 & V_1/v_1 & U_3/u_2 & V_3/v_2 \\ 19628 & 29442 & -19628 & -29442 \\ 29442 & 44163 & -29442 & -44163 \\ -19628 & -29442 & 19628 & 29442 \\ -29442 & -44163 & 29442 & 44163 \end{array}$$

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ 19628 & 29442 & 0 & 0 & -19628 & -29442 \\ 29442 & 44163 & 0 & 0 & -29442 & -44163 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -19628 & -29442 & 0 & 0 & 19628 & 29442 \\ -29442 & -44163 & 0 & 0 & 29442 & 44163 \end{array}$$

Introduction to FEA: Analysis Procedures

5- Apply Boundary Conditions

Governing equation, must be modified to account for the boundary conditions, is a set of simultaneous algebraic/ordinary differential/partial differential equations that can be written in expanded matrix form.

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

$$\begin{bmatrix} 134628 & 29442 & 0 & \vdots & -115000 & -19628 & -29442 \\ 29442 & 44163 & 0 & \vdots & 0 & -29442 & -44163 \\ 0 & 0 & 76666.67 & \vdots & 0 & 0 & -76666.67 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -115000 & 0 & 0 & \vdots & 115000 & 0 & 0 \\ -19628 & -29442 & 0 & \vdots & 0 & 19628 & 29442 \\ -29442 & -44163 & -76666.67 & \vdots & 0 & 29442 & 120829.67 \end{bmatrix} \begin{Bmatrix} U_1 = 0 \\ V_1 = 0 \\ V_2 = 0 \\ \cdots \\ U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} R_{X1} \\ R_{Y1} \\ R_{Y2} \\ \cdots \\ 0 \\ 12000 \\ 0 \end{Bmatrix}$$

Introduction to FEA: Analysis Procedures

6- Solve for the unknown nodal displacements

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

It should be mentioned that K will always have an inverse for well-posed problems solved by the finite element method.

Introduction to FEA: Analysis Procedures

6-Calculation of the Element Resultants

SUPPORT REACTIONS

$$\{F_P\} = [K_{PF}]\{\delta_F\} \qquad \begin{Bmatrix} R_{X1} \\ R_{Y1} \\ R_{Y2} \end{Bmatrix} = \begin{bmatrix} -115000 & -19628 & -29442 \\ 0 & -29442 & -44163 \\ 0 & 0 & -76666.67 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.9635 \\ -0.2348 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -18 \\ 18 \end{Bmatrix} \text{ kN}$$

MEMBERS' FORCES

Once all the displacements are known, the member forces can be easily obtained

$$\{\delta\} \longrightarrow \{\overline{d_3}\} \longrightarrow \{d_3\} = [C_3]^T\{\overline{d_3}\}$$

$$\{f_3\} = \begin{bmatrix} 63791.43 & 0 & -63791.43 & 0 \\ 0 & 0 & 0 & 0 \\ -63791.43 & 0 & 63791.43 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.3391 \\ -0.9319 \end{Bmatrix} = \begin{Bmatrix} -21.631 \\ 0 \\ 21.631 \\ 0 \end{Bmatrix} \text{ kN}$$

Introduction to FEA: Analysis Procedures

Static Problem
(ODEs or PDEs)

FEM →

System of Algebraic Equations
(Linear or Non-linear)

$$\frac{d}{dx}\left(AE\frac{du(x)}{dx}\right) = w(x)$$

$[K]\{a\} = f$ → $\{a\}$

Dynamic Problem
(PDEs)

FEM →

System of ODEs
(Linear or Non-linear)

$$\frac{d}{dx}\left(AE\frac{\partial u(x,t)}{\partial x}\right) - w(x,t) = \rho\frac{\partial^2 u(x,t)}{\partial t^2}$$

$[M]\{\ddot{a}(t)\} + [K]\{a(t)\} = f(t)$ → $\{a\}$

Introduction to FEA: Analysis Procedures

Non-linear Structural Problems

- **Material Nonlinearity:** Due to non-linear constitutive law (e.g., polymer materials)
- **Geometric Nonlinearity:** Due to Large displacements or large rotations
- **Boundary Nonlinearity:** Due to non-linearity of boundary conditions (i.e., contact problems)

Introduction to FEA: Analysis Procedures

FEM

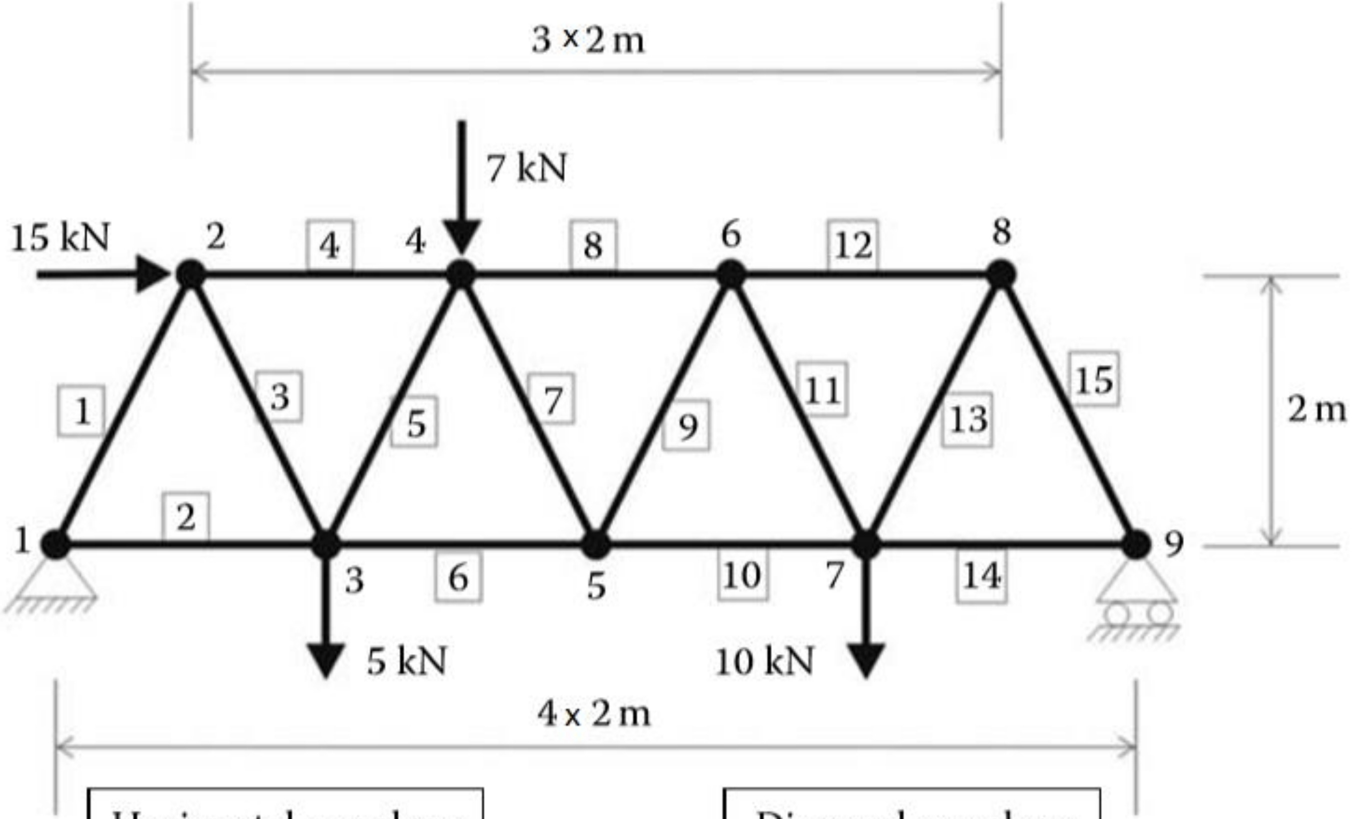
- **Implicit Method**
  - **Static Problems**
    - **Linear Problems:** $[K]\{a\} = f$ — Linear system of algebraic equations
    - **Non-Linear Problems:** $[K(\{a\})]\{a\} = f(\{a\})$ — Non-linear sys. of algebraic equations
  - **Dynamic Problems (Dynamic Implicit)**
    - **Linear Problems:** $[M(t)]\{\ddot{a}(t)\} + [K(t)]\{a(t)\} = f(t)$ — Linear sys. of ODEs
    - **Non-Linear Problems:** $[M(t,\{a\})]\{\ddot{a}(t)\} + [K(t,\{a\})]\{a(t)\} = f(t,\{a\})$ — Non-linear sys. of ODEs
- **Explicit Method** →
  - **Quasi-Static Problems**
  - **Dynamic Problems**

  → $[M(t,\{a\})]\{\ddot{a}(t)\} + [K(t,\{a\})]\{a(t)\} = f(t,\{a\})$

  Non-linear or Linear sys. of ODEs

Problem 1: Truss Problem

Problem Discerption



Horizontal members
$E = 30 \times 10^6$ kN/m^2
$A = 0.045$ m^2

Diagonal members
$E = 30 \times 10^6$ kN/m^2
$A = 0.02$ m^2

Problem 1: Truss Problem

All input and output data must be specified in consistent units

Quantity	SI	SI (mm)	US Unit (ft)	US Unit (inch)
Length	m	mm	ft	in
Force	N	N	lbf	lbf
Mass	kg	tonne (10^3 kg)	slug	lbf s^2/in
Time	s	s	s	s
Stress	Pa (N/m^2)	MPa (N/mm^2)	lbf/ft^2	psi (lbf/in^2)
Energy	J	mJ (10^{-3} J)	ft lbf	in lbf
Density	kg/m^3	tonne/mm^3	slug/ft^3	lbf s^2/in^4

Problem 1: Truss Problem

Data Preparation (Create Input file)

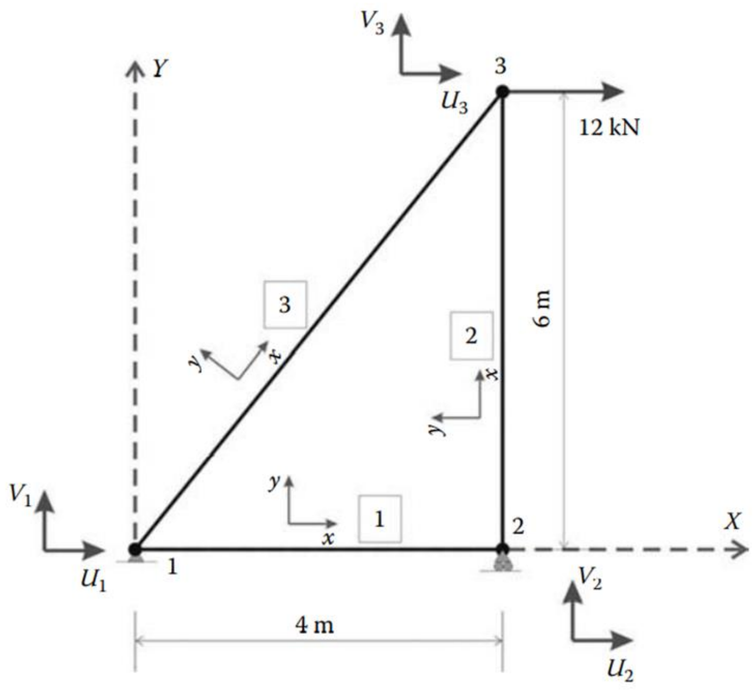
Nodes Coordinates → $\mathbf{geom} = \begin{bmatrix} 0 & 0 \\ 4000 & 0 \\ 4000 & 6000 \end{bmatrix}$

Element Connectivity → $\mathbf{connec} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}$

Material and Geometrical Properties → $\mathbf{prop} = \begin{bmatrix} 200000 & 2300 \\ 200000 & 2300 \\ 200000 & 2300 \end{bmatrix}$

Boundary Conditions → $\mathbf{nf} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ → $\mathbf{nf} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$

Loading → $\mathbf{load} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1200 & 0 \end{bmatrix}$



Problem 1: Truss Problem

Discretization and Interpolation

$$u(x) = c_0 + c_1 x$$

$$\begin{cases} u(x=0) = u_1 = c_0 \\ u(x=L) = u_2 = c_0 + c_1 L \end{cases} \Rightarrow u(x) = \left[\frac{(u_2 - u_1)}{L}\right] x + u_1$$

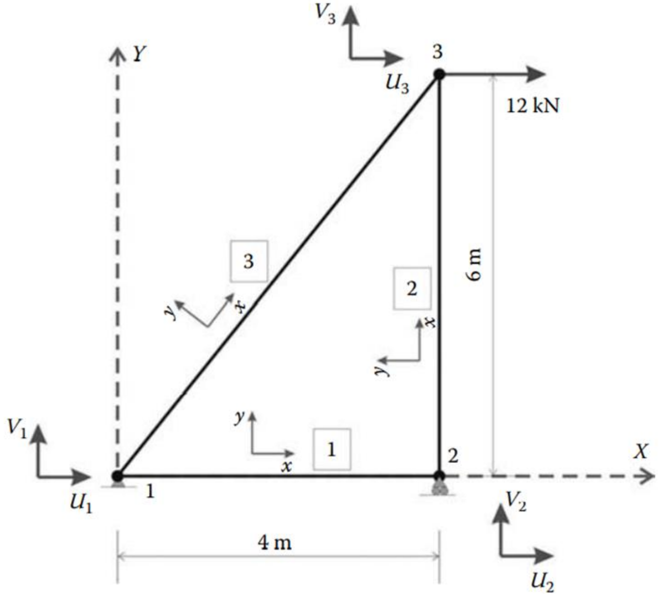
$$v(x) = c'_0 + c'_1 x$$

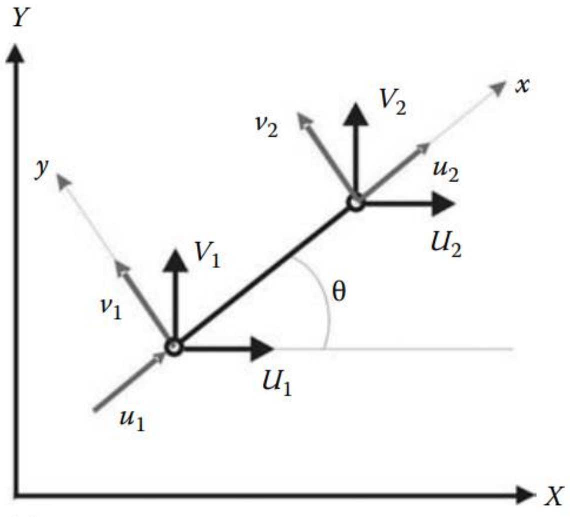
$$\begin{cases} v(x=0) = v_1 = c'_0 \\ v(x=L) = v_2 = c'_0 + c'_1 L \end{cases} \Rightarrow v(x) = \left[\frac{(v_2 - v_1)}{L}\right] x + v_1$$

$$\begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = [N]\{d_e\}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}$$

$$N_1 = \left(1 - \frac{x}{L}\right) \qquad N_2 = \frac{x}{L}$$





$$\{d_e\} = \{u_1, v_1, u_2, v_2\}^T$$

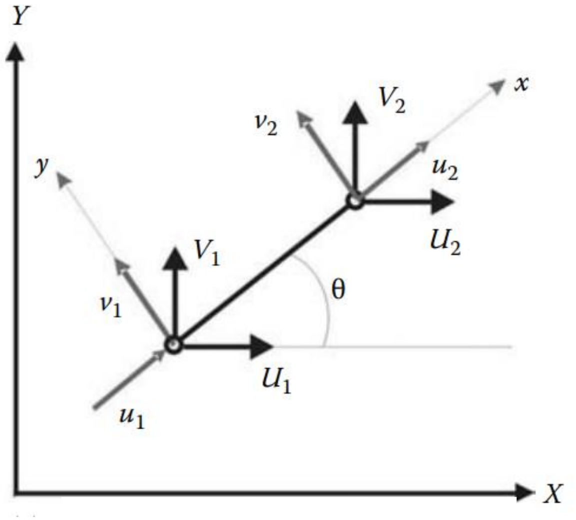
Problem 1: Truss Problem

Direct Approach

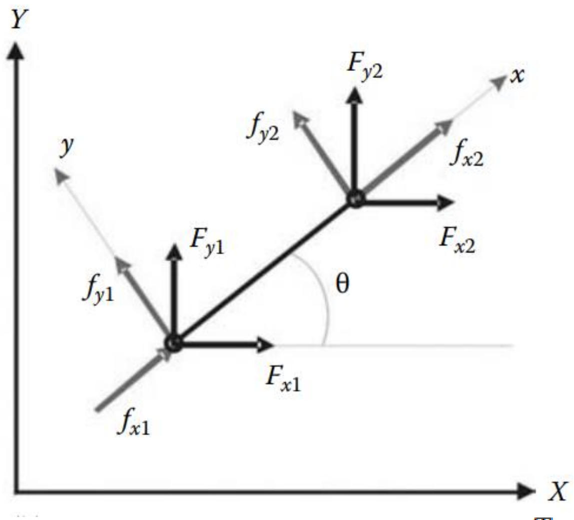
$$\begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = [N]\{d_e\} \Rightarrow \{\varepsilon\} = [L][N]\{d_e\}$$

$$\{\sigma\} = [D]\{\varepsilon\} \Rightarrow \{\sigma\} = [D][L][N]\{d_e\}$$

$$\sigma_x = E\varepsilon_x \quad \begin{cases} f_{x1} = EA\left(\dfrac{u_1 - u_2}{L}\right) \\ f_{x2} = EA\left(\dfrac{u_2 - u_1}{L}\right) \end{cases}$$



$$\{d_e\} = \{u_1, v_1, u_2, v_2\}^T$$
$$\{\bar{d}_e\} = \{U_1, V_1, U_2, V_2\}^T$$



$$\{f_e\} = \{f_{x1}, f_{y1}, f_{x2}, f_{y2}\}^T$$
$$\{\bar{f}_e\} = \{F_{x1}, F_{y1}, F_{x2}, F_{y2}\}^T$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \qquad N_1 = \left(1 - \frac{x}{L}\right) \qquad N_2 = \frac{x}{L}$$

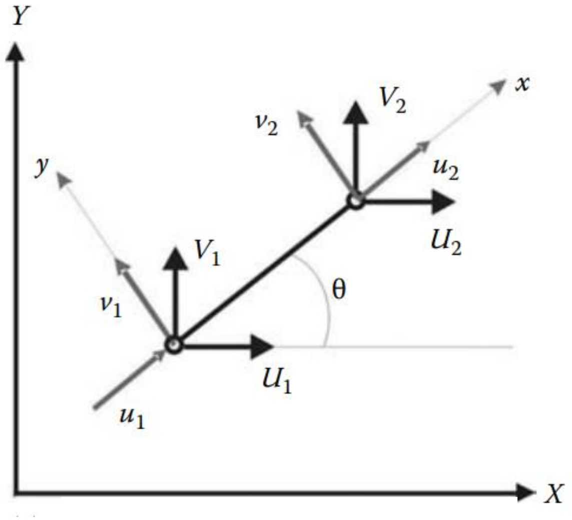
$$[L] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \end{bmatrix} \qquad [D] = [E]$$

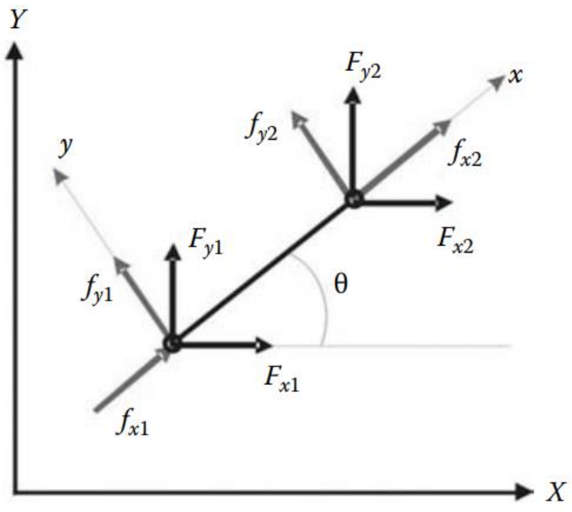
Problem 1: Truss Problem

Local Stiffness Matrix

$$\begin{bmatrix} \frac{AE}{L} & 0 & -\frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & \frac{AE}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix}$$

$$[K_e]\{d_e\} = \{f_e\}$$





$$\{d_e\} = \{u_1, v_1, u_2, v_2\}^T$$

$$\{\bar{d}_e\} = \{U_1, V_1, U_2, V_2\}^T$$

$$\{f_e\} = \{f_{x1}, f_{y1}, f_{x2}, f_{y2}\}^T$$

$$\{\bar{f}_e\} = \{F_{x1}, F_{y1}, F_{x2}, F_{y2}\}^T$$

$$[C] = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & cos(\theta) & -sin(\theta) \\ 0 & 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

$$\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Problem 1: Truss Problem

Global Stiffness Matrix

$$[K_e]\{d_e\} = \{f_e\}$$

$$\{d_e\} = [C]^T\{\overline{d}_e\}$$

$$\{f_e\} = [C]^T\{\overline{f}_e\}$$

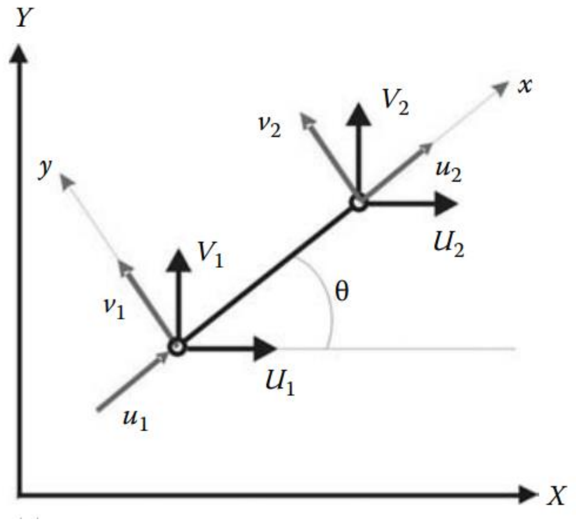
$$[C][K_e][C]^T\{\overline{d}_e\} = \{\overline{f}_e\}$$

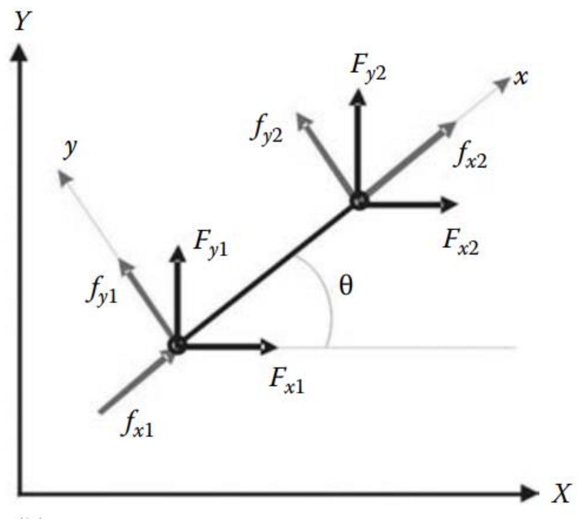
$$[\overline{K}_e] = [C][K_e][C]^T$$

$$[\overline{K}_e]\{\overline{d}_e\} = \{\overline{f}_e\}$$

Element stiffness matrix in the global coordinate system

$$[C] = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & cos(\theta) & -sin(\theta) \\ 0 & 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$





$$\{d_e\} = \{u_1, v_1, u_2, v_2\}^T$$

$$\{\bar{d}_e\} = \{U_1, V_1, U_2, V_2\}^T$$

$$\{f_e\} = \{f_{x1}, f_{y1}, f_{x2}, f_{y2}\}^T$$

$$\{\overline{f}_e\} = \{F_{x1}, F_{y1}, F_{x2}, F_{y2}\}^T$$

Problem 1: Truss Problem

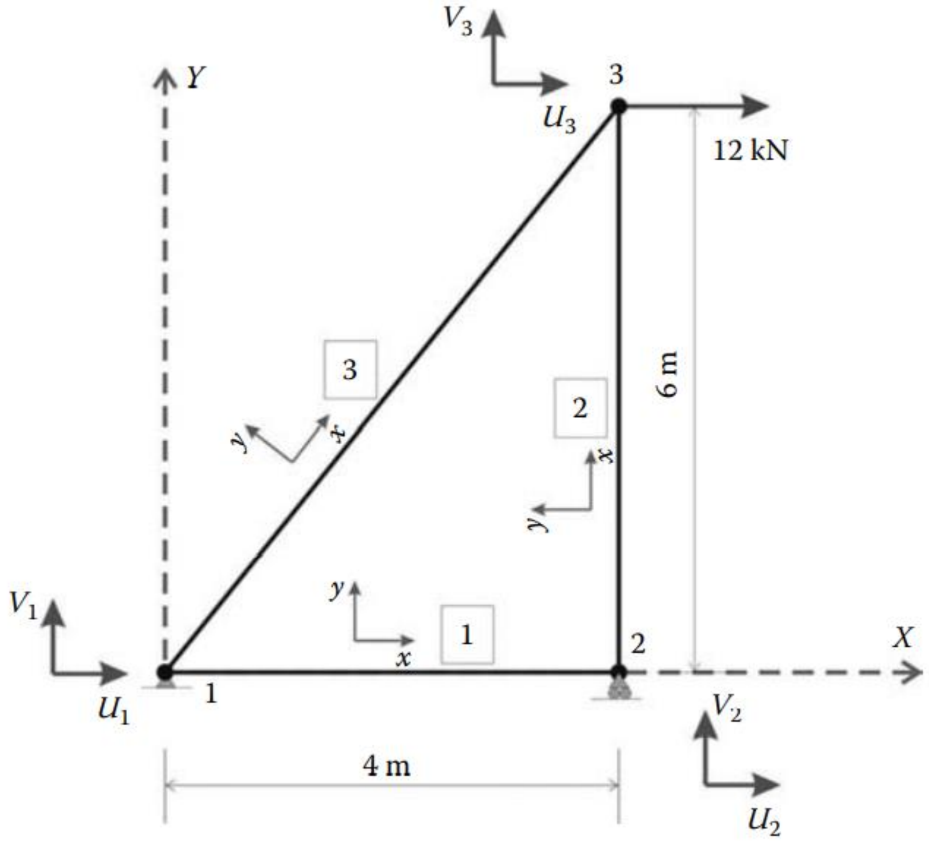
Assemblage

The individual element nodal equilibrium equations are assembled into the global nodal equilibrium equations.

$$[K_1]_L = \begin{bmatrix} 115000 & 0 & -115000 & 0 \\ 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_2]_L = \begin{bmatrix} 76666.67 & 0 & -76666.67 & 0 \\ 0 & 0 & 0 & 0 \\ -76666.67 & 0 & 76666.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_3]_L = \begin{bmatrix} 63791.43 & 0 & -63791.43 & 0 \\ 0 & 0 & 0 & 0 \\ -63791.43 & 0 & 63791.43 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Problem 1: Truss Problem

Assemblage

$$[K_1]_L = \begin{bmatrix} 115000 & 0 & -115000 & 0 \\ 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[C_1] = \begin{bmatrix} \cos(0) & -\sin(0) & 0 & 0 \\ \sin(0) & \cos(0) & 0 & 0 \\ 0 & 0 & \cos(0) & -\sin(0) \\ 0 & 0 & \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K_1]_G = \begin{matrix} & U_1/u_1 & V_1/v_1 & U_2/u_2 & V_2/v_2 \\ U_1/u_1 & 115000 & 0 & -115000 & 0 \\ V_1/v_1 & 0 & 0 & 0 & 0 \\ U_2/u_2 & -115000 & 0 & 115000 & 0 \\ V_2/v_2 & 0 & 0 & 0 & 0 \end{matrix}$$

$$[K_2]_L = \begin{bmatrix} 76666.67 & 0 & -76666.67 & 0 \\ 0 & 0 & 0 & 0 \\ -76666.67 & 0 & 76666.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[C_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[K_2]_G = \begin{matrix} U_2/u_2 \\ V_2/v_2 \\ U_3/u_3 \\ V_3/v_3 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 76666.67 & 0 & -76666.67 \\ 0 & 0 & 0 & 0 \\ 0 & -76666.67 & 0 & 76666.67 \end{bmatrix}$$

$$[K_3]_L = \begin{bmatrix} 63791.43 & 0 & -63791.43 & 0 \\ 0 & 0 & 0 & 0 \\ -63791.43 & 0 & 63791.43 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[C_3] = \begin{bmatrix} 0.554699 & -0.832051 & 0 & 0 \\ 0.832051 & 0.554699 & 0 & 0 \\ 0 & 0 & 0.554699 & -0.832051 \\ 0 & 0 & 0.832051 & 0.554699 \end{bmatrix}$$

$$[K_3]_G = \begin{matrix} & U_1/u_1 & V_1/v_1 & U_3/u_2 & V_3/v_2 \\ U_1/u_1 & 19628 & 29442 & -19628 & -29442 \\ V_1/v_1 & 29442 & 44163 & -29442 & -44163 \\ U_3/u_2 & -19628 & -29442 & 19628 & 29442 \\ V_3/v_2 & -29442 & -44163 & 29442 & 44163 \end{matrix}$$

Problem 1: Truss Problem

Assemblage

$$[K_1]_G = \begin{array}{c|cccc} & U_1/u_1 & V_1/v_1 & U_2/u_2 & V_2/v_2 \\ \hline U_1/u_1 & 115000 & 0 & -115000 & 0 \\ V_1/v_1 & 0 & 0 & 0 & 0 \\ U_2/u_2 & -115000 & 0 & 115000 & 0 \\ V_2/v_2 & 0 & 0 & 0 & 0 \end{array}$$

$$[\mathbf{K}] = \begin{array}{c|cccccc} & U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline U_1 & 115000 & 0 & -115000 & 0 & 0 & 0 \\ V_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_2 & -115000 & 0 & 115000 & 0 & 0 & 0 \\ V_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$[K_2]_G = \begin{array}{c} U_2/u_2 \\ V_2/v_2 \\ U_3/u_3 \\ V_3/v_3 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 76666.67 & 0 & -76666.67 \\ 0 & 0 & 0 & 0 \\ 0 & -76666.67 & 0 & 76666.67 \end{bmatrix}$$

$$[\mathbf{K}] = \begin{array}{c|cccccc} & U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline U_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_2 & 0 & 0 & 0 & 76666.67 & 0 & -76666.67 \\ U_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_3 & 0 & 0 & 0 & -76666.67 & 0 & 76666.67 \end{array}$$

$$[K_3]_G = \begin{array}{c|cccc} & U_1/u_1 & V_1/v_1 & U_3/u_2 & V_3/v_2 \\ \hline U_1/u_1 & 19628 & 29442 & -19628 & -29442 \\ V_1/v_1 & 29442 & 44163 & -29442 & -44163 \\ U_3/u_2 & -19628 & -29442 & 19628 & 29442 \\ V_3/v_2 & -29442 & -44163 & 29442 & 44163 \end{array}$$

$$[\mathbf{K}] = \begin{array}{c|cccccc} & U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline U_1 & 19628 & 29442 & 0 & 0 & -19628 & -29442 \\ V_1 & 29442 & 44163 & 0 & 0 & -29442 & -44163 \\ U_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_3 & -19628 & -29442 & 0 & 0 & 19628 & 29442 \\ V_3 & -29442 & -44163 & 0 & 0 & 29442 & 44163 \end{array}$$

Problem 1: Truss Problem

Assemblage

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline 115000 & 0 & -115000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -115000 & 0 & 115000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Element 1

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 76666.67 & 0 & -76666.67 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -76666.67 & 0 & 76666.67 \end{array}$$

Element 2

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline 19628 & 29442 & 0 & 0 & -19628 & -29442 \\ 29442 & 44163 & 0 & 0 & -29442 & -44163 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -19628 & -29442 & 0 & 0 & 19628 & 29442 \\ -29442 & -44163 & 0 & 0 & 29442 & 44163 \end{array}$$

Element 3

$$[\mathbf{K}] = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \\ \hline 115000+19628 & 29442 & -115000 & 0 & -19628 & -29442 \\ 29442 & 44163 & 0 & 0 & -29442 & -44163 \\ -115000 & 0 & 115000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 76666.67 & 0 & -76666.67 \\ -19628 & -29442 & 0 & 0 & 19628 & 29442 \\ -29442 & -44163 & 0 & -76666.67 & 29442 & 44163+76666.67 \end{array}$$

Problem 1: Truss Problem

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

→

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

→

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Problem 1: Truss Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$\{F_P\} = [K_{PF}]\{\delta_F\} \qquad \begin{Bmatrix} R_{X1} \\ R_{Y1} \\ R_{Y2} \end{Bmatrix} = \begin{bmatrix} -115000 & -19628 & -29442 \\ 0 & -29442 & -44163 \\ 0 & 0 & -76666.67 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.9635 \\ -0.2348 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -18 \\ 18 \end{Bmatrix} \text{kN}$$

MEMBERS' FORCES

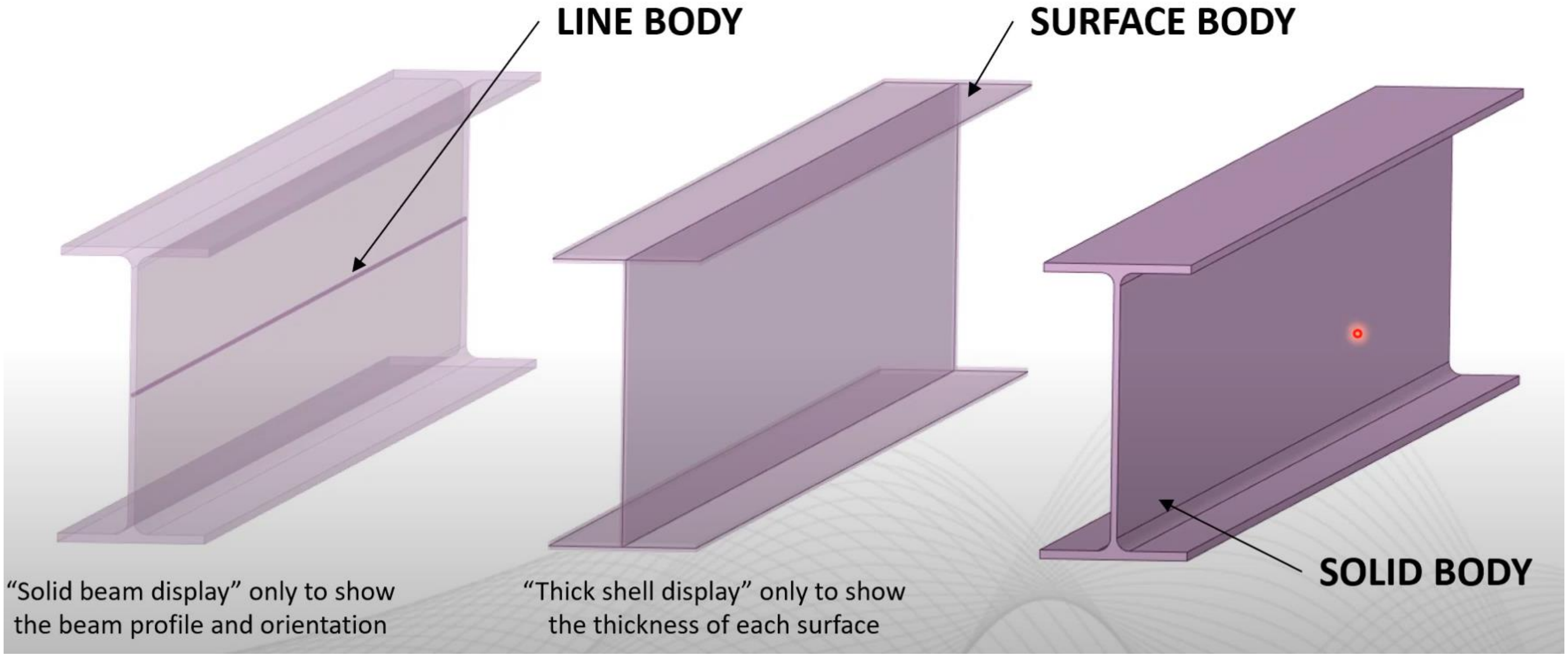
Once all the displacements are known, the member forces can be easily obtained

$$\{\delta\} \longrightarrow \{\overline{d_3}\} \longrightarrow \{d_3\} = [C_3]^T\{\overline{d_3}\}$$

$$\{f_3\} = \begin{bmatrix} 63791.43 & 0 & -63791.43 & 0 \\ 0 & 0 & 0 & 0 \\ -63791.43 & 0 & 63791.43 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.3391 \\ -0.9319 \end{Bmatrix} = \begin{Bmatrix} -21.631 \\ 0 \\ 21.631 \\ 0 \end{Bmatrix} \text{kN}$$

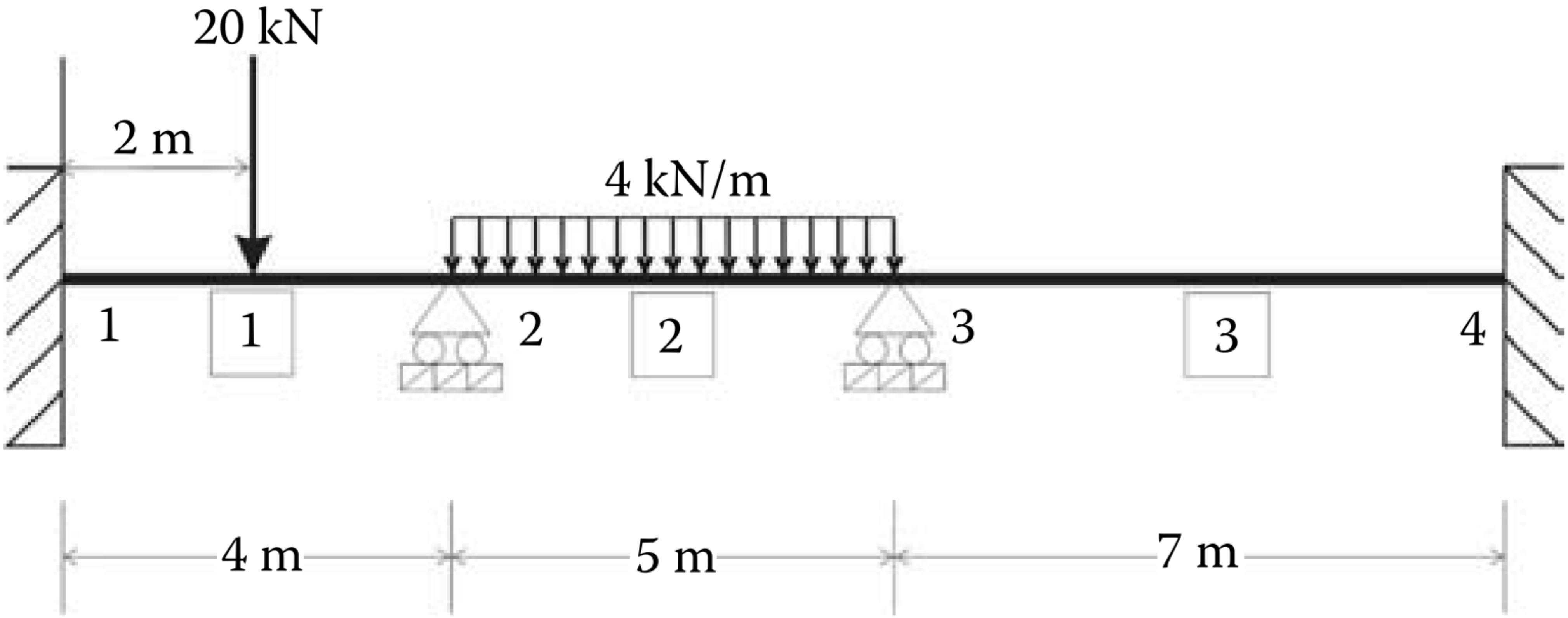
Problem 2: Beam Problem

Different types of modeling and associated assumptions



Problem 2: Beam Problem

Problem Discerption



$E = 200\ 000$ MPa, $I = 200 \times 10^6$ mm^4

Problem 2: Beam Problem

Data Preparation (Create Input file)

Nodes Coordinates

$$\textbf{geom} = \begin{bmatrix} 0 \\ 4000 \\ 9000 \\ 16000 \end{bmatrix}$$

Element Connectivity

$$\textbf{connec} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Material and Geometrical Properties

$$\textbf{prop} = \begin{bmatrix} 200000 & 200.e+6 \\ 200000 & 200.e+6 \\ 200000 & 200.e+6 \end{bmatrix}$$

Boundary Conditions

$$\textbf{nf} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \textbf{nf} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Loading

Element	F_{y1}	M_1	F_{y2}	M_2
1	-10^4	-10^7	-10^4	10^7
2	-10^4	-8.33×10^6	-10^4	-8.33×10^6
3	0	0	0	0

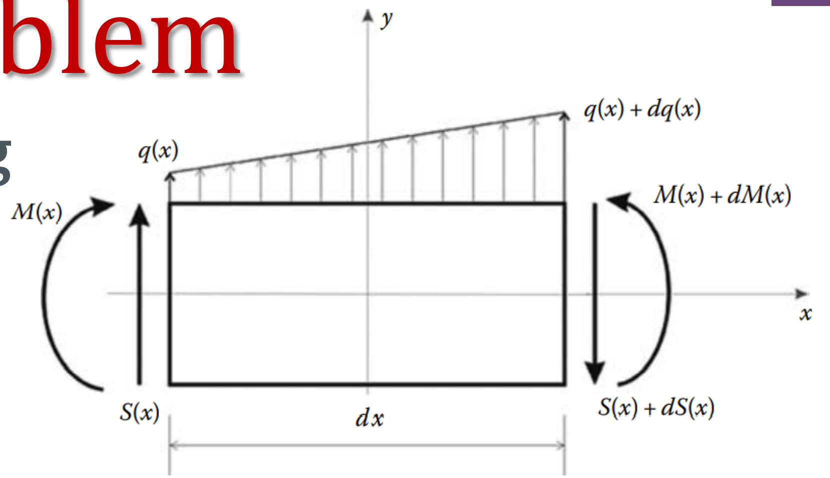
Problem 2: Beam Problem

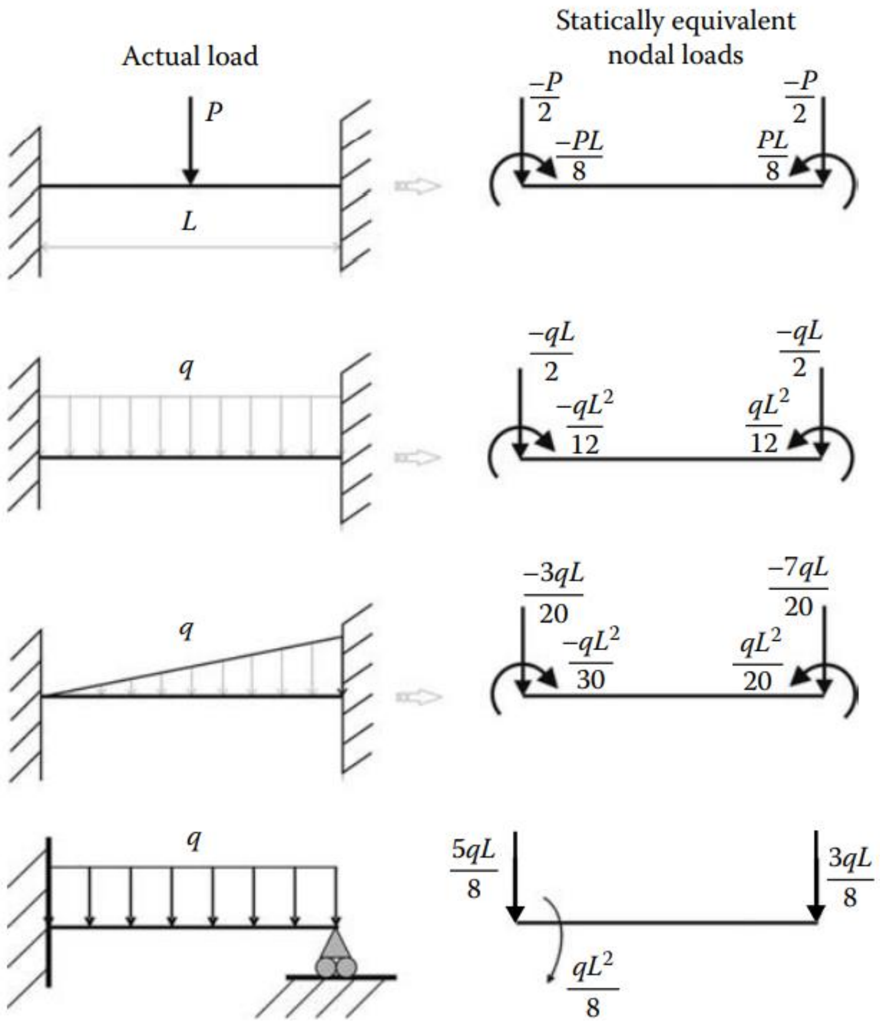
Euler–Bernoulli theory of bending

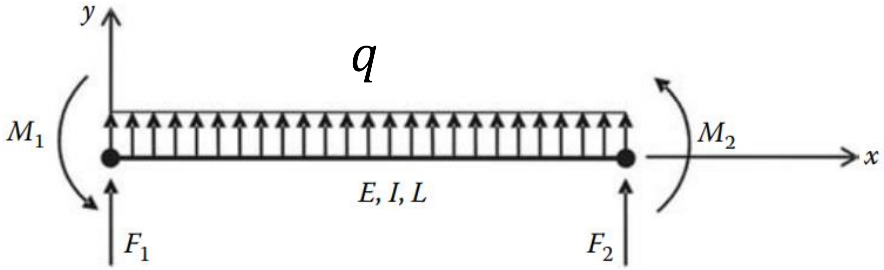
$$\frac{d^2w}{dx^2} = \frac{M}{EI}$$

$$\frac{d^3w}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{S}{EI}$$

$$\frac{d^4w}{dx^4} = \frac{1}{EI}\frac{dS}{dx} = \frac{q(x)}{EI}$$

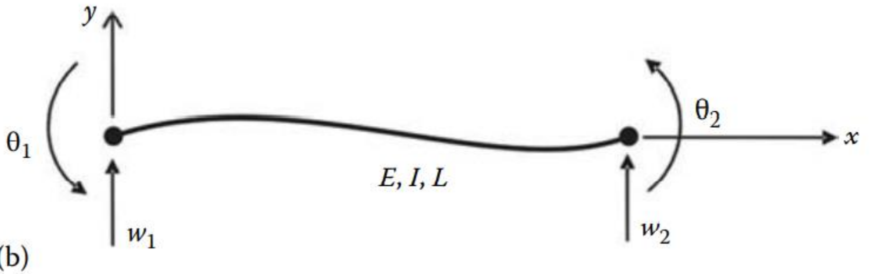






$$\{F_e\} = \{F_1, M_1, F_2, M_2\}^T$$

$$\{F_e\} = \begin{Bmatrix} -\dfrac{qL}{2} \\ -\dfrac{qL^2}{12} \\ -\dfrac{qL}{2} \\ +\dfrac{qL^2}{12} \end{Bmatrix}$$



(b)

$$\{d_e\} = \{w_1, \theta_1, w_2, \theta_2\}^T$$

Problem 2: Beam Problem

Interpolation (Shape Function)

$$
\begin{cases}
w(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4 \\
w(x=0) = w_1 = c_4 \\
\left.\dfrac{dw}{dx}\right|_{x=0} = \theta_1 = c_3 \\
w(x=L) = w_2 = c_1 L^3 + c_2 L^2 + c_3 L + c_4 \\
\left.\dfrac{dw}{dx}\right|_{x=L} = \theta_2 = 3c_1 L^2 + 2c_2 L + c_3
\end{cases}
$$

$$
w(x) = \left[\frac{2}{L^3}(w_1 - w_2) + \frac{1}{L^2}(\theta_1 + \theta_2)\right]x^3 + \left[-\frac{3}{L^2}(w_1 - w_2) - \frac{1}{L}(2\theta_1 + \theta_2)\right]x^2 + \theta_1 x + w_1
$$

$$w(x) = [N]\{d_e\}$$

$$[w(x)] = [N]\{d_e\}$$

$1\times1 \quad 1\times4 \quad 4\times1$

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

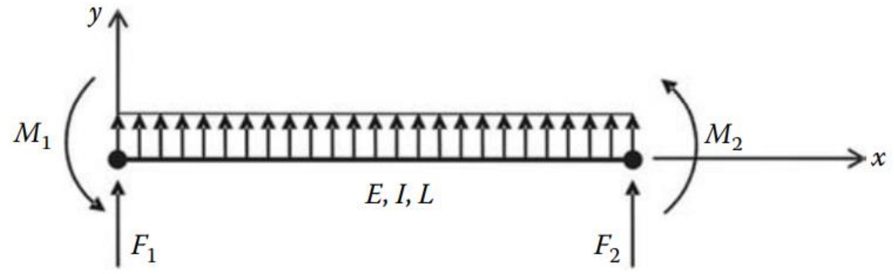
$$N_1 = \frac{1}{L^3}(2x^3 - 3x^2 L + L^3) \qquad N_2 = \frac{1}{L^3}(x^3 L - 2x^2 L^2 + x L^3)$$

$$N_3 = \frac{1}{L^3}(-2x^3 + 3x^2 L) \qquad N_4 = \frac{1}{L^3}(x^3 L - x^2 L^2)$$

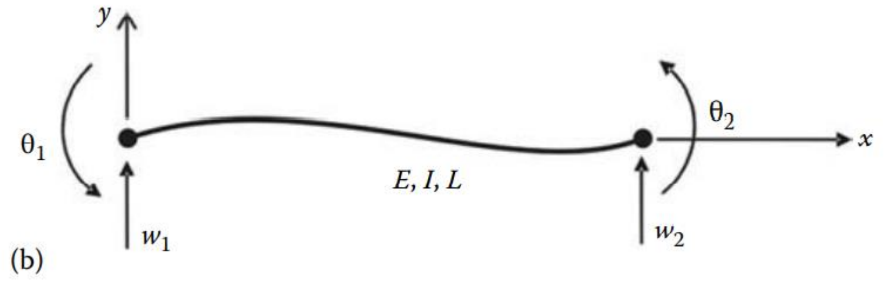
Problem 2: Beam Problem

Direct Equilibrium Approach

$$
\begin{cases}
F_1 = EI\dfrac{d^3w(x)}{dx^3}\Big|_{x=0} = \dfrac{EI}{L^3}(12w_1 + 6L\theta_1 - 12w_2 + 6L\theta_2) \\
M_1 = -EI\dfrac{d^2w(x)}{dx^2}\Big|_{x=0} = \dfrac{EI}{L^3}(6Lw_1 + 4L^2\theta_1 - 6Lw_2 + 2L^2\theta_2) \\
F_2 = -EI\dfrac{d^3w(x)}{dx^3}\Big|_{x=L} = \dfrac{EI}{L^3}(-12w_1 - 6L\theta_1 + 12w_2 - 6L\theta_2) \\
M_2 = EI\dfrac{d^2w(x)}{dx^2}\Big|_{x=L} = \dfrac{EI}{L^3}(6Lw_1 + 2L^2\theta_1 - 6Lw_2 + 4L^2\theta_2)
\end{cases}
$$



$$\{F_e\} = \{F_1, M_1, F_2, M_2\}^T$$



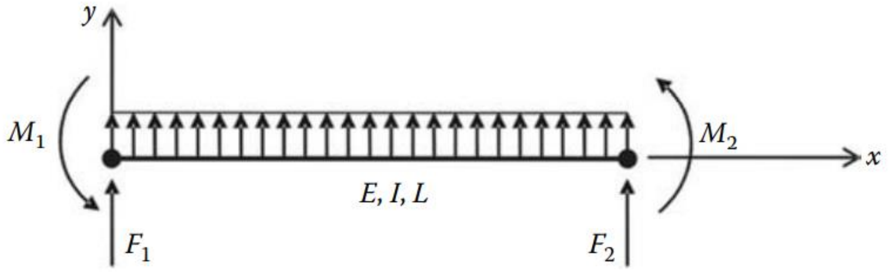
$$\{d_e\} = \{w_1, \theta_1, w_2, \theta_2\}^T$$

Problem 2: Beam Problem

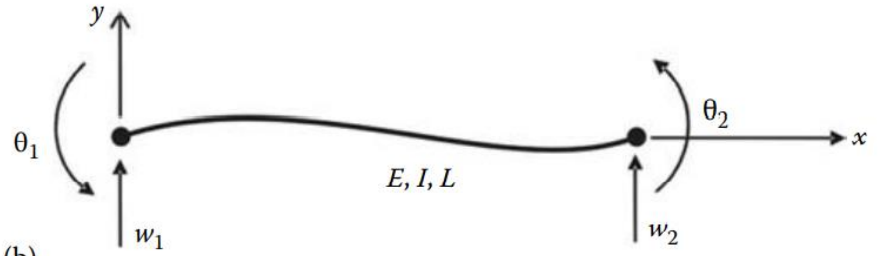
Local Stiffness Matrix

$$
\begin{Bmatrix} F_{y1} \\ M_1 \\ F_{y2} \\ M_2 \end{Bmatrix} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}
$$

$$\{f_e\} = [K_e]\{\delta_e\}$$



$$\{F_e\} = \{F_1, M_1, F_2, M_2\}^T$$



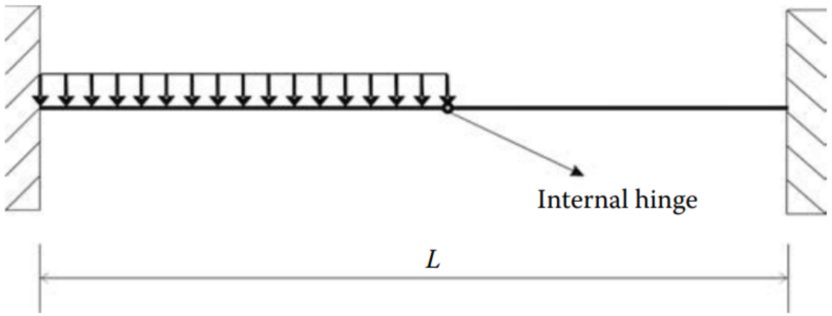
(b)

$$\{d_e\} = \{w_1, \theta_1, w_2, \theta_2\}^T$$

Problem 2: Beam Problem

Local Stiffness Matrix: Internal Hinge

Internal Hinge
- Discontinuity in the slope of the deflection curve
- Zero value of the bending moment



Procedure

Discretize the beam using two elements

The hinge should be accounted for only once; either associated with element 1 or with element 2

If the beam is discretized with two elements, one with a hinge at its right end and the other with a hinge at its left, the result will be a singular stiffness matrix.

$$\begin{bmatrix} 3EI/L^3 & 3EI/L^2 & -3EI/L^3 & 0 \\ 3EI/L^2 & 3EI/L & -3EI/L^2 & 0 \\ -3EI/L^3 & -3EI/L^2 & 3EI/L^3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_{11} \\ \theta_{11} \\ w_{12} \\ \theta_{12} \end{Bmatrix} = \begin{Bmatrix} F_{11} \\ M_{11} \\ F_{12} \\ M_{12} \end{Bmatrix}$$

$$\begin{bmatrix} 3EI/L^3 & 0 & -3EI/L^3 & 3EI/L^2 \\ 0 & 0 & 0 & 0 \\ -3EI/L^3 & 0 & 3EI/L^3 & -3EI/L^2 \\ 3EI/L^2 & 0 & -3EI/L^2 & 3EI/L \end{bmatrix} \begin{Bmatrix} w_{21} \\ \theta_{21} \\ w_{22} \\ \theta_{22} \end{Bmatrix} = \begin{Bmatrix} F_{21} \\ M_{21} \\ F_{22} \\ M_{22} \end{Bmatrix}$$

Problem 2: Beam Problem

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix} \Rightarrow$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\Rightarrow \{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Problem 2: Beam Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\} \quad \xrightarrow{\text{If } \{\delta_p\} = 0} \quad \{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

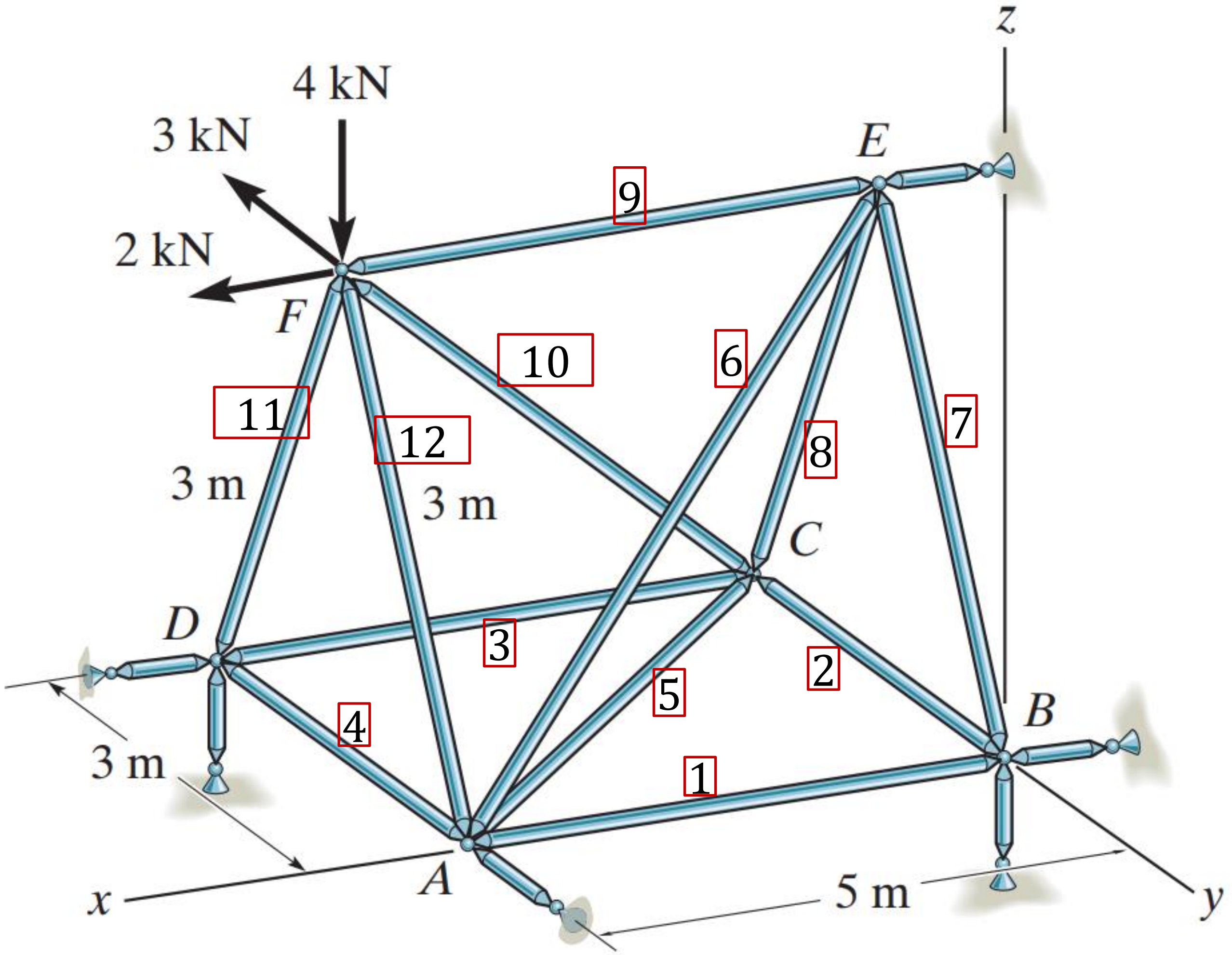
Once all the displacements are known, the member forces can be easily obtained

$$\{\delta\} \longrightarrow \{d_e\} \longrightarrow \{F_e\} = [K_e]\{d_e\} - \{F_0\}$$

$\{F_e\}$: The vector of equivalent nodal forces at element level

3D Truss Problem

Problem Discerption



$E = 200\ GPa \quad A = 0.02\ m^2$

3D Truss Problem

Consistent Units

All input and output data must be specified in consistent units

Quantity	SI	SI (mm)	US Unit (ft)	US Unit (inch)
Length	m	mm	ft	in
Force	N	N	lbf	lbf
Mass	kg	tonne (10^3 kg)	slug	lbf s^2/in
Time	s	s	s	s
Stress	Pa (N/m^2)	MPa (N/mm^2)	lbf/ft^2	psi (lbf/in^2)
Energy	J	mJ (10^{-3} J)	ft lbf	in lbf
Density	kg/m^3	tonne/mm^3	slug/ft^3	lbf s^2/in^4

3D Truss Problem

Data Preparation (Create Input file)

Nodes Coordinates	geom (nnd, dim=3)
Element Connectivity	connec (nel, nne=2)
Material and Geometrical Properties	$E = 200\ GPa$ $A = 0.02\ m^2$
Boundary Conditions	nf (nnd, nodof=3)
Loading	load (nnd, dim=3)

3D Truss Problem

Discretization and Interpolation

$$u(x) = c_0 + c_1 x$$

$$\begin{cases} u(x=0) = u_1 = c_0 \\ u(x=L) = u_2 = c_0 + c_1 L \end{cases} \Rightarrow u(x) = \left[\frac{(u_2 - u_1)}{L}\right] x + u_1$$

$$v(x) = c'_0 + c'_1 x$$

$$\begin{cases} v(x=0) = v_1 = c'_0 \\ v(x=L) = v_2 = c'_0 + c'_1 L \end{cases} \Rightarrow v(x) = \left[\frac{(v_2 - v_1)}{L}\right] x + v_1$$

$$w(x) = c''_0 + c''_1 x$$

$$\begin{cases} w(x=0) = w_1 = c''_0 \\ w(x=L) = w_2 = c''_0 + c''_1 L \end{cases} \Rightarrow w(x) = \left[\frac{(w_2 - w_1)}{L}\right] x + w_1$$

$$\Rightarrow \begin{Bmatrix} u(x) \\ v(x) \\ w(x) \end{Bmatrix} = [N]\{d_e\}$$

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix}$$

$$N_1 = \left(1 - \frac{x}{L}\right) \qquad N_2 = \frac{x}{L}$$

$$\{d_e\} = \{u_1\ v_1\ w_1\ u_2\ v_2\ w_2\}^T$$

3D Truss Problem

Local Stiffness Matrix

$$\begin{Bmatrix} u(x) \\ v(x) \\ w(x) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

$$N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

$$\varepsilon_{xx} = \frac{\partial u(x)}{\partial x} \rightarrow L = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{xx} = E\varepsilon_{xx} \rightarrow D = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$K_e = \int_0^L B^T DB\,Adx = \frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

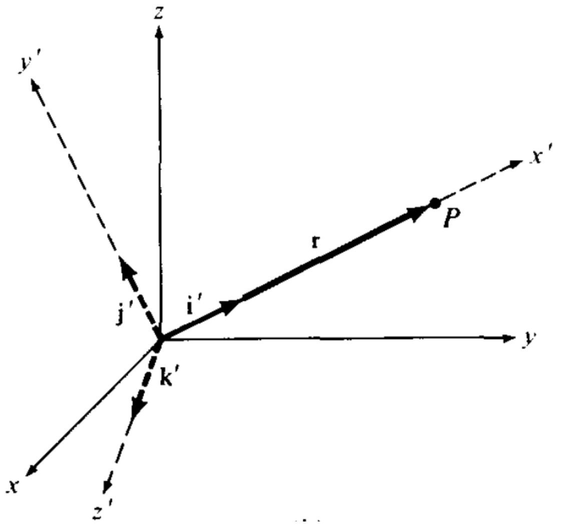
$$\begin{cases} f_{1x} = EA\left(\dfrac{u_1 - u_2}{L}\right) \\ f_{2x} = EA\left(\dfrac{u_2 - u_1}{L}\right) \end{cases} \Rightarrow \frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \\ f_{x2} \\ f_{y2} \\ f_{z2} \end{Bmatrix} \Rightarrow [K_e]\{d_e\} = \{f_e\}$$

3D Truss Problem

Transformation Matrix

$$r = r_x i + r_y j + r_z k = r'_x i' + r'_y j' + r'_z k'$$

$$\begin{cases} r_x i.i' + r_y j.i' + r_z k.i' = r'_x \\ r_x i.j' + r_y j.j' + r_z k.j' = r'_y \\ r_x i.k' + r_y j.k' + r_z k.k' = r'_z k' \end{cases}$$



$$\begin{Bmatrix} r'_x \\ r'_y \\ r'_z \end{Bmatrix} = \begin{bmatrix} i.i' & j.i' & k.i' \\ i.j' & j.j' & k.j' \\ i.k' & j.k' & k.k' \end{bmatrix} \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} \Rightarrow \begin{Bmatrix} r'_x \\ r'_y \\ r'_z \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos(x,x') & \cos(y,x') & \cos(z,x') \\ \cos(x,y') & \cos(y,y') & \cos(z,y') \\ \cos(x,z') & \cos(y,z') & \cos(z,z') \end{bmatrix}}_{[T]} \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix}$$

$$[R] = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}$$

$$\cos(x,x') = \frac{x_j - x_i}{L} \qquad \cos(y,x') = \frac{y_j - y_i}{L} \qquad \cos(z,x') = \frac{z_j - z_i}{L}$$

$$D = \sqrt{\cos^2(x,x') + \cos^2(y,x')}$$

$$\cos(x,y') = \frac{\cos(y,x')}{D} \qquad \cos(y,y') = -\frac{\cos(x,x')}{D} \qquad \cos(z,y') = -\frac{\cos(x,x')}{D}$$

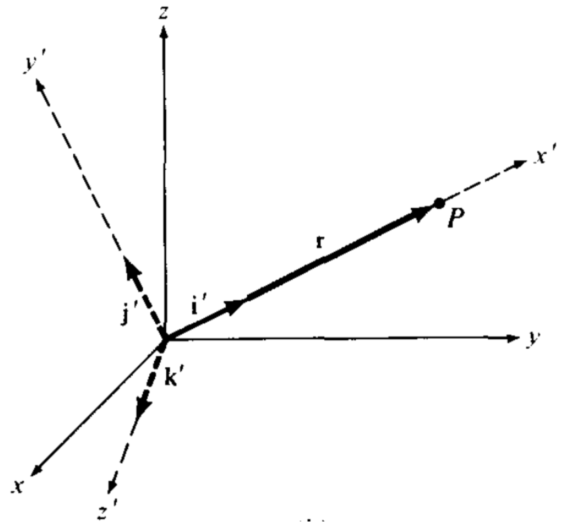
$$\cos(x,z') = -\frac{\cos(x,x')\cos(z,x')}{D} \qquad \cos(y,z') = -\frac{\cos(y,x')\cos(z,x')}{D} \qquad \cos(z,z') = D$$

3D Truss Problem

Transformation Matrix

$$e'_i = T_{ij}\, e_j$$

$$\begin{Bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} e'_1 . e_1 & e'_1 . e_2 & e'_1 . e_3 \\ e'_2 . e_1 & e'_2 . e_2 & e'_2 . e_3 \\ e'_3 . e_1 & e'_3 . e_2 & e'_3 . e_3 \end{bmatrix}}_{[T]} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix}$$



$$[R] = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}$$

3D Truss Problem

Transformation Matrix

Element stiffness matrix in the global coordinate system

Matrix Form

$$[k] = [R]^T [k^/][R]$$

Index Form

$$\{k\} = k_{ij} e_i e_j$$

$$\{k'\} = k'_{mn} e'_m e'_n$$

$$e'_m = r_{mi} e_i$$

$$e'_n = r_{nj} e_j$$

$$k_{ij} = k'_{mn} \, r_{mi} r_{nj}$$

3D Truss Problem

More Efficient Procedure

$$\begin{cases} f_{1x} = EA\left(\dfrac{u_1 - u_2}{L}\right) \\ f_{2x} = EA\left(\dfrac{u_2 - u_1}{L}\right) \end{cases} \Rightarrow [k_e] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{r'_x\} = [\cos(x,y') \quad \cos(y,y') \quad \cos(z,y')]\{r_x\}$$

$$[K_e] = [R]^T[k_e][R]$$

$$\cos(x,x') = \frac{x_j - x_i}{L} \qquad \cos(y,x') = \frac{y_j - y_i}{L} \qquad \cos(z,x') = \frac{z_j - z_i}{L}$$

$$[R] = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} = \begin{bmatrix} \frac{x_j - x_i}{L} & \frac{y_j - y_i}{L} & \frac{z_j - z_i}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_j - x_i}{L} & \frac{y_j - y_i}{L} & \frac{z_j - z_i}{L} \end{bmatrix}$$

$[R]$ × Global Coordinate = Local Coordinate

$[R]^T$ × Local Coordinate = Global Coordinate

3D Truss Problem

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix} \Rightarrow$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\Rightarrow \{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

3D Truss Problem

Calculation of the Element Resultants

MEMBERS' FORCES

Once all the displacements are known, the member forces can be easily obtained

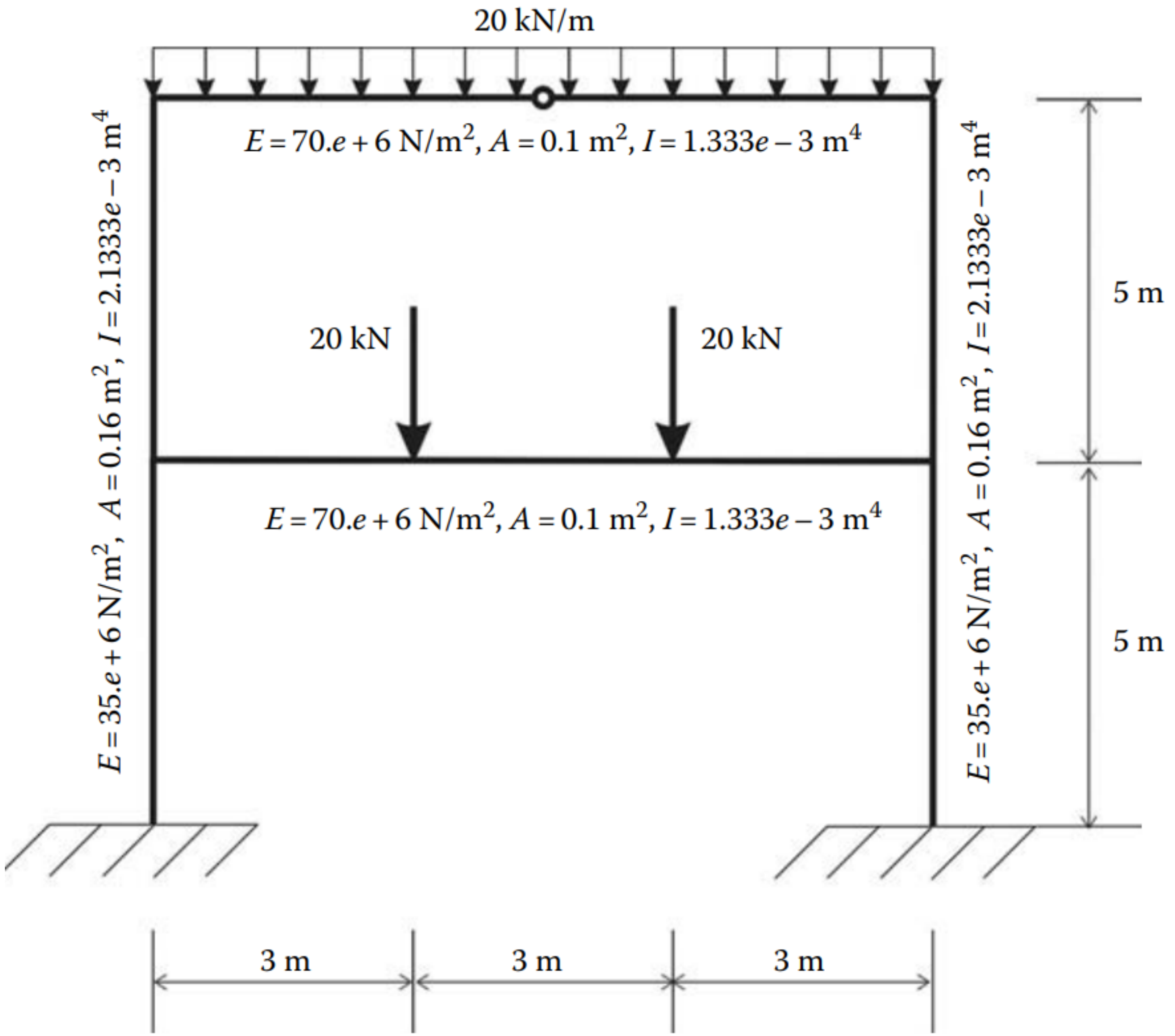
$$\{\delta\} \Rightarrow \text{Element Displacement in Global Coordinate} \times \begin{bmatrix} \frac{x_j - x_i}{L} & \frac{y_j - y_i}{L} & \frac{z_j - z_i}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_j - x_i}{L} & \frac{y_j - y_i}{L} & \frac{z_j - z_i}{L} \end{bmatrix} = \text{Element Displacement in Local Coordinate}$$

$\Downarrow$

Member Force

Problem 4: 2D Frames

Problem Description



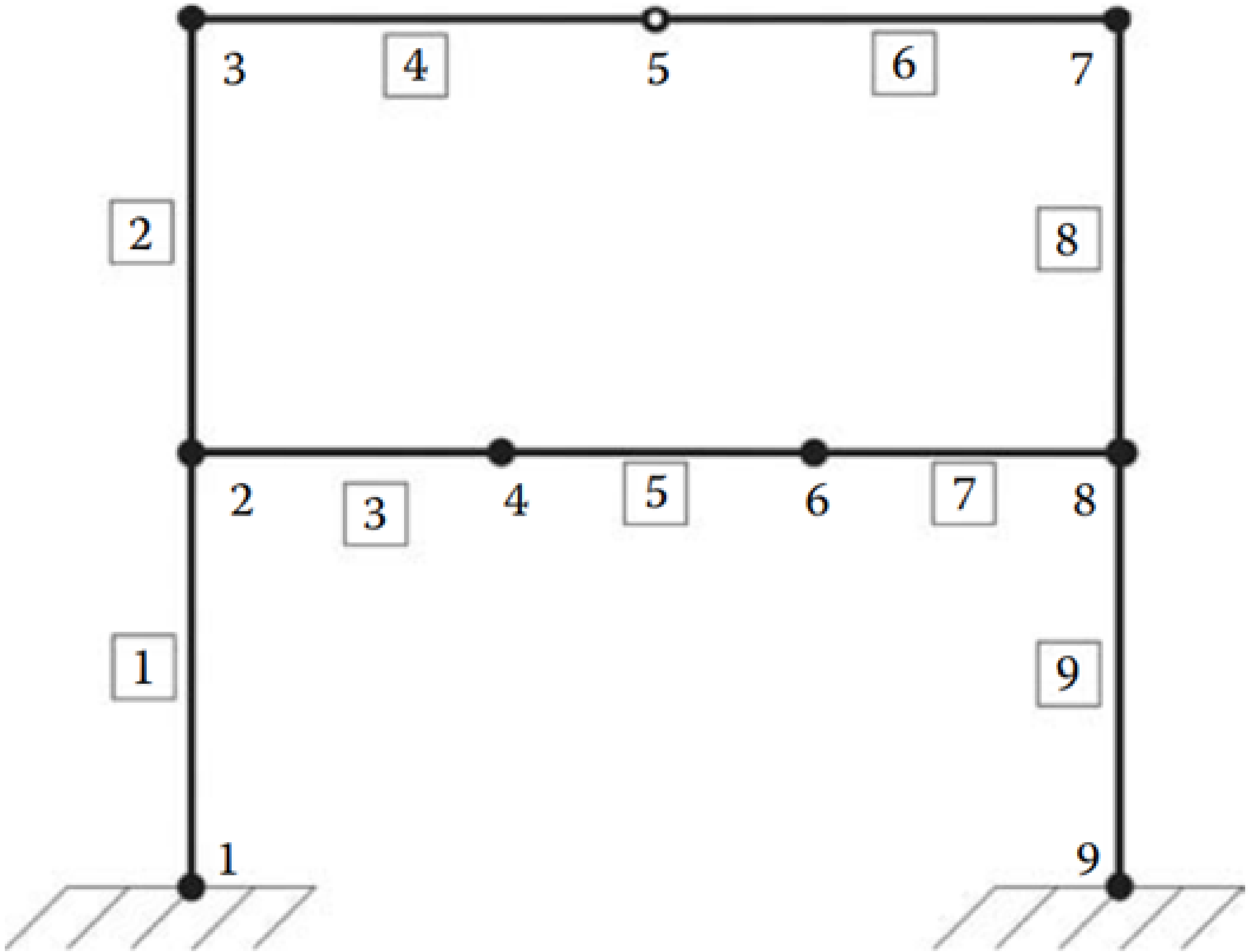
Problem 4: 2D Frames

All input and output data must be specified in consistent units

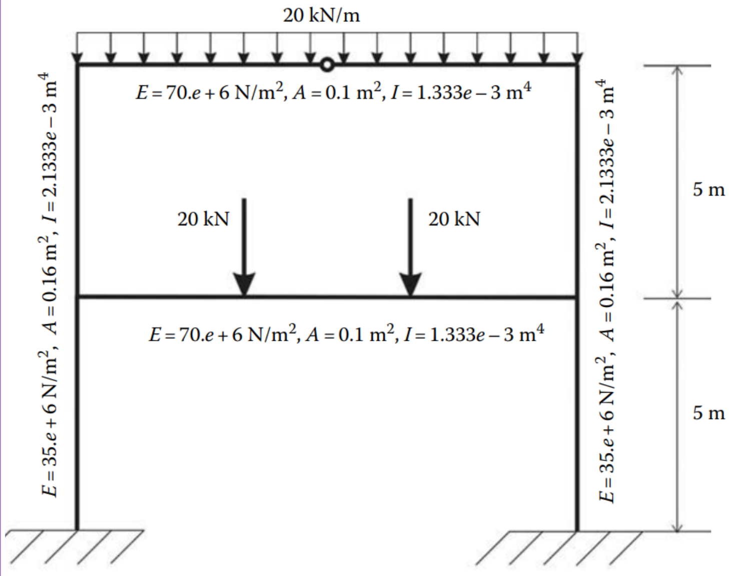
Quantity	SI	SI (mm)	US Unit (ft)	US Unit (inch)
Length	m	mm	ft	in
Force	N	N	lbf	lbf
Mass	kg	tonne (10^3 kg)	slug	lbf s^2/in
Time	s	s	s	s
Stress	Pa (N/m^2)	MPa (N/mm^2)	lbf/ft^2	psi (lbf/in^2)
Energy	J	mJ (10^{-3} J)	ft lbf	in lbf
Density	kg/m^3	$tonne/mm^3$	$slug/ft^3$	lbf s^2/in^4

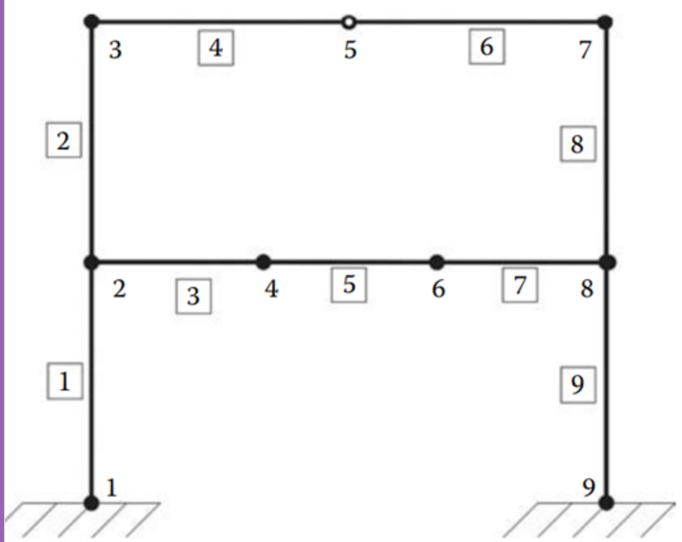
Problem 4: 2D Frames

Discretization

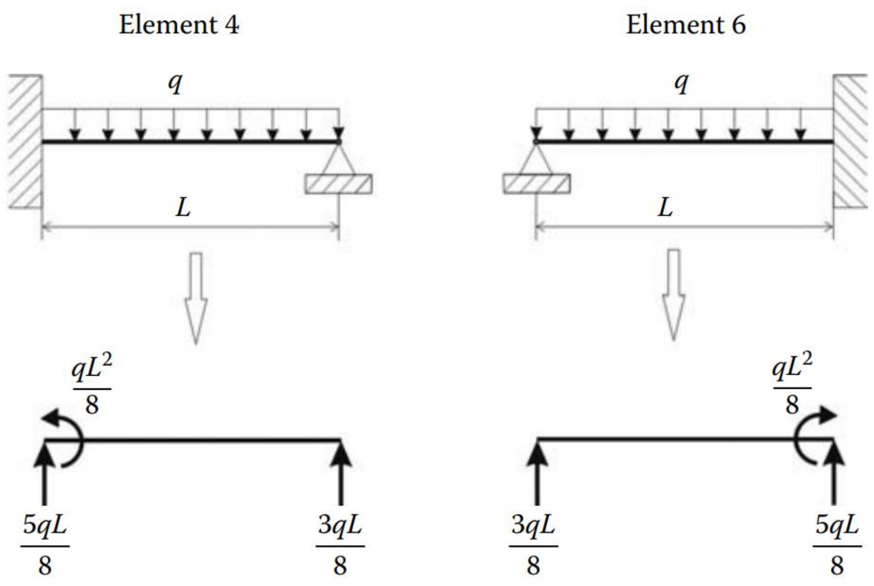


Problem 4: 2D Frames





Statically Equivalent Nodal Loads



Problem 4: 2D Frames

Data Preparation (Create Input file)

Nodes Coordinates	geom (nnd, dim=2)
Element Connectivity	connec (nel, nne=2)
Material and Geometrical Properties	E A I
Boundary Conditions	nf (nnd, nodof=3)
Loading	load (nnd, nodof=3)

Problem 4: 2D Frames

Interpolation (Shape Function)

$$v(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

$$\begin{cases} v(x=0) = v_1 = c_0 \\ \left.\dfrac{dv}{dx}\right|_{x=0} = \theta_1 = c_1 \\ v(x=L) = v_2 = c_3L^3 + c_2L^2 + c_1L + c_0 \\ \left.\dfrac{dv}{dx}\right|_{x=L} = \theta_2 = 3c_3L^2 + 2c_2L + c_1 \end{cases}$$

$$\Rightarrow v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_1 + \theta_2)\right]x^3 + \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\theta_1 + \theta_2)\right]x^2 + \theta_1 x + v_1$$

$$\Downarrow$$

$$v(x) = \frac{1}{L^3}[2x^3 - 3x^2L + L^3]v_1 + \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]\theta_1 + \frac{1}{L^3}[-2x^3 + 3x^2L]v_2 + \frac{1}{L^3}[x^3L - x^2L^2]\theta_2$$

$$u(x) = c_1x + c_0$$

$$\begin{cases} u(x=0) = u_1 = c_0 \\ u(x=L) = u_2 = c_0 + c_1L \end{cases} \Rightarrow u(x) = \left[\frac{(u_2 - u_1)}{L}\right]x + u_1 \Rightarrow u(x) = \left[1 - \frac{x}{L}\right]u_1 + \left[\frac{x}{L}\right]u_2$$

Problem 4: 2D Frames

Interpolation (Shape Function)

$$u(x) = \left[1 - \frac{x}{L}\right] u_1 + \left[\frac{x}{L}\right] u_2 \quad \Rightarrow \quad N_1 = \left[1 - \frac{x}{L}\right] \quad N_2 = \left[\frac{x}{L}\right]$$

$$v(x) = \frac{1}{L^3}[2x^3 - 3x^2L + L^3]v_1 + \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]\theta_1 + \frac{1}{L^3}[-2x^3 + 3x^2L]v_2 + \frac{1}{L^3}[x^3L - x^2L^2]\theta_2$$

$$N_3 = \frac{1}{L^3}[2x^3 - 3x^2L + L^3] \quad N_4 = \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3] \quad N_5 = \frac{1}{L^3}[-2x^3 + 3x^2L] \quad N_6 = \frac{1}{L^3}[x^3L - x^2L^2]$$

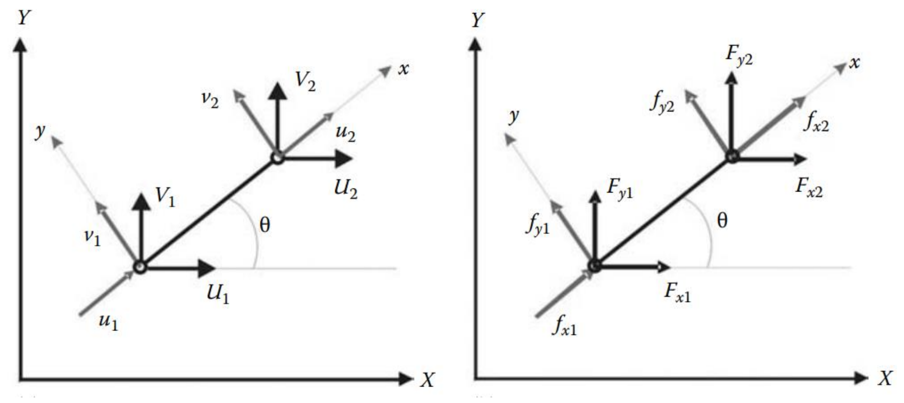
$$\begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = [N]\{d_e\} \quad \Rightarrow \quad \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_3 & N_4 & 0 & N_5 & N_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

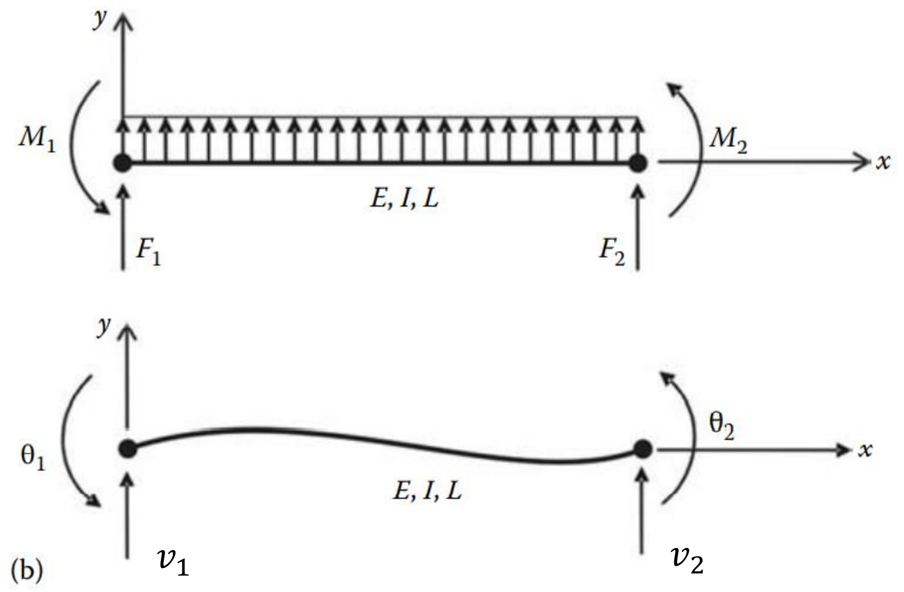
Problem 4: 2D Frames

Direct Equilibrium Approach

$$\begin{cases} f_{x1} = EA\left(\dfrac{u_1 - u_2}{L}\right) \\ f_{x2} = EA\left(\dfrac{u_2 - u_1}{L}\right) \end{cases}$$

$$\begin{cases} F_{y1} = EI\dfrac{d^3v(x)}{dx^3}\Big|_{x=0} = \dfrac{EI}{L^3}(12v_1 + 6L\theta_1 - 12v_2 + 6L\theta_2) \\ M_1 = -EI\dfrac{d^2v(x)}{dx^2}\Big|_{x=0} = \dfrac{EI}{L^3}(6Lv_1 + 4L^2\theta_1 - 6Lv_2 + 2L^2\theta_2) \\ F_{y2} = -EI\dfrac{d^3v(x)}{dx^3}\Big|_{x=L} = \dfrac{EI}{L^3}(-12v_1 - 6L\theta_1 + 12v_2 - 6L\theta_2) \\ M_2 = EI\dfrac{d^2v(x)}{dx^2}\Big|_{x=L} = \dfrac{EI}{L^3}(6Lv_1 + 2L^2\theta_1 - 6Lv_2 + 4L^2\theta_2) \end{cases}$$



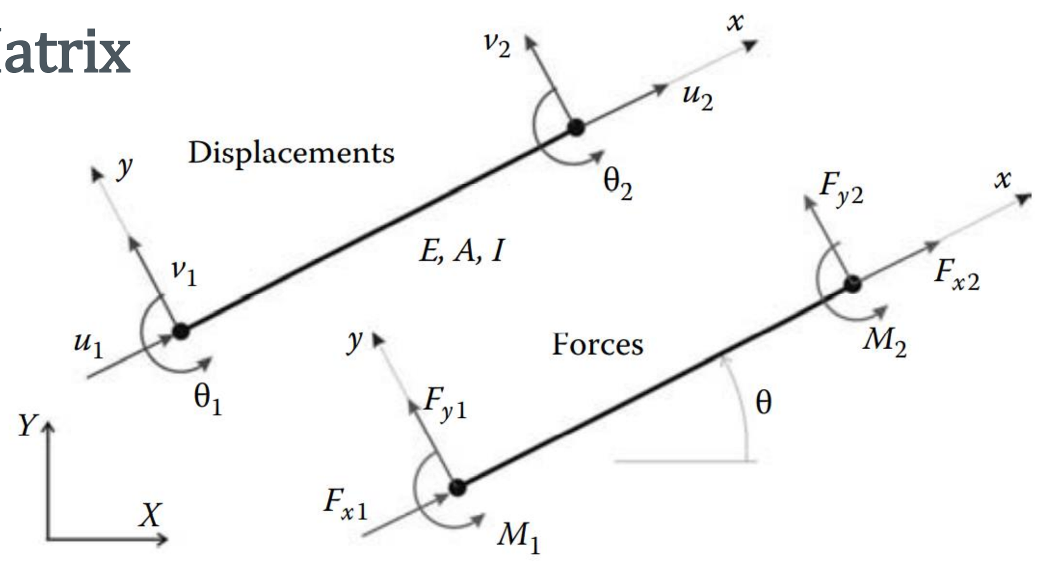


Problem 4: 2D Frames

Local Stiffness Matrix

$$[K_e] = \int_0^L [B]^T [D][B] A dx$$

$$[K_e] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$



$$\{d_e\} = \{u_1, v_1, \theta_1, u_2, v_2, \theta_2\}^T \quad \{F_e\} = \{F_{x1}, F_{y1}, M_1, F_{x2}, F_{y2}, M_2\}^T$$

hinge at its right end:

$$[K_e] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 3EI/L^3 & 3EI/L^2 & 0 & -3EI/L^3 & 0 \\ 0 & 3EI/L^2 & 3EI/L & 0 & -3EI/L^2 & 0 \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -3EI/L^3 & -3EI/L^2 & 0 & 3EI/L^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

hinge at its left end:

$$[K_e] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 3EI/L^3 & 0 & 0 & -3EI/L^3 & 3EI/L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -3EI/L^3 & 0 & 0 & 3EI/L^3 & -3EI/L^2 \\ 0 & 3EI/L^2 & 0 & 0 & -3EI/L^2 & 3EI/L \end{bmatrix}$$

Problem 4: 2D Frames

Global Stiffness Matrix

$$[K_e]\{d_e\} = \{f_e\}$$

$$\{d_e\} = [C]^T\{\overline{d_e}\}$$

$$\{f_e\} = [C]^T\{\overline{f_e}\}$$

$$[C][K_e][C]^T\{\overline{d_e}\} = \{\overline{f_e}\} \quad \Rightarrow \quad [\overline{K_e}]\{\overline{d_e}\} = \{\overline{f_e}\}$$

$$[\overline{K_e}] = [C][K_e][C]^T$$

$R^T k_e R \rightarrow R = \begin{bmatrix} T & \\ & T \end{bmatrix}$

$$[C] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Element stiffness matrix in the global coordinate system

$$[\overline{K_e}] = [C][K_e][C]^T$$

Assemblage

Problem 4: 2D Frames

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

$K_{FF}\,\delta_F = F_F$

linsolve

K \ F

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Problem 4: 2D Frames

Calculation of the Element Resultants

SUPPORT REACTIONS

$$\{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

Once all the displacements are known, the member forces can be easily obtained

$$\{\delta\} \longrightarrow \{\overline{d_3}\} \longrightarrow \{d_3\} = [C_3]^T\{\overline{d_3}\}$$

$K\delta = F$

$k_e d_e = F_e$

$F_e - F_0$

3D Frame Problem

Problem Description

3D Frame Problem

Discretization

3D Frame Problem

Statically Equivalent Nodal Loads

3D Frame Problem

Data Preparation (Create Input file)

Nodes Coordinates geom (nnd, dim = 3)

Element Connectivity connec (nel, nne = 2)

Material and Geometrical Properties

$E = 200\ GPa$
$A = 0.02\ m^2$
$I = \ \ m^4$

Boundary Conditions nf (nnd, nodof = 6)

Loading load (nnd, nodof = 6)

3D Frame Problem

Interpolation (Shape Function)

$$v(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

$$\begin{cases} v(x=0) = v_1 = c_0 \\ \left.\dfrac{dv}{dx}\right|_{x=0} = \theta_1 = c_1 \\ v(x=L) = v_2 = c_3L^3 + c_2L^2 + c_1L + c_0 \\ \left.\dfrac{dv}{dx}\right|_{x=L} = \theta_2 = 3c_3L^2 + 2c_2L + c_1 \end{cases}$$

$$v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_1 + \theta_2)\right]x^3 + \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\theta_1 + \theta_2)\right]x^2 + \theta_1 x + v_1$$

$$v(x) = \frac{1}{L^3}[2x^3 - 3x^2L + L^3]v_1 + \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]\theta_1 + \frac{1}{L^3}[-2x^3 + 3x^2L]v_2 + \frac{1}{L^3}[x^3L - x^2L^2]\theta_2$$

$$w(x) = \frac{1}{L^3}[2x^3 - 3x^2L + L^3]w_1 + \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]\theta_1 + \frac{1}{L^3}[-2x^3 + 3x^2L]w_2 + \frac{1}{L^3}[x^3L - x^2L^2]\theta_2$$

$$u(x) = c_1x + c_0$$

$$\begin{cases} u(x=0) = u_1 = c_0 \\ u(x=L) = u_2 = c_0 + c_1L \end{cases}$$

$$u(x) = \left[\frac{(u_2 - u_1)}{L}\right]x + u_1$$

$$u(x) = \left[1 - \frac{x}{L}\right]u_1 + \left[\frac{x}{L}\right]u_2$$

3D Frame Problem

Interpolation (Shape Function)

$$u(x) = \left[1 - \frac{x}{L}\right] u_1 + \left[\frac{x}{L}\right] u_2 \quad \Rightarrow \quad N_1 = \left[1 - \frac{x}{L}\right] \quad N_2 = \left[\frac{x}{L}\right]$$

$$v(x) = \frac{1}{L^3}[2x^3 - 3x^2L + L^3]v_1 + \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]\theta_1 + \frac{1}{L^3}[-2x^3 + 3x^2L]v_2 + \frac{1}{L^3}[x^3L - x^2L^2]\theta_2$$

$$N_3 = \frac{1}{L^3}[2x^3 - 3x^2L + L^3] \quad N_4 = \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3] \quad N_5 = \frac{1}{L^3}[-2x^3 + 3x^2L] \quad N_6 = \frac{1}{L^3}[x^3L - x^2L^2]$$

$$\begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = [N]\{d_e\} \quad \Rightarrow \quad \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_3 & N_4 & 0 & N_4 & N_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

3D Frame Problem

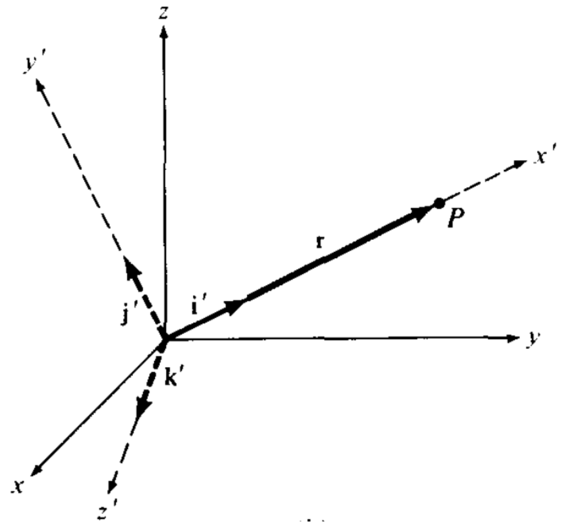
Local Stiffness Matrix

$$[K_e] = \int_0^L [B]^T [D][B] A dx$$

$$[K_e] = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}$$

3D Frames

Transformation Matrix



$$r = r_x i + r_y j + r_z k = r'_x i' + r'_y j' + r'_z k' \quad \begin{cases} r_x i.i' + r_y j.i' + r_z k.i' = r'_x \\ r_x i.j' + r_y j.j' + r_z k.j' = r'_y \\ r_x i.k' + r_y j.k' + r_z k.k' = r'_z k' \end{cases}$$

$$\begin{Bmatrix} r'_x \\ r'_y \\ r'_z \end{Bmatrix} = \begin{bmatrix} i.i' & j.i' & k.i' \\ i.j' & j.j' & k.j' \\ i.k' & j.k' & k.k' \end{bmatrix} \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} \Rightarrow \begin{Bmatrix} r'_x \\ r'_y \\ r'_z \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos(x,x') & \cos(y,x') & \cos(z,x') \\ \cos(x,y') & \cos(y,y') & \cos(z,y') \\ \cos(x,z') & \cos(y,z') & \cos(z,z') \end{bmatrix}}_{[T]} \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix}$$

$$[T] \Rightarrow [R] = \begin{bmatrix} [T] & 0 & 0 & 0 \\ 0 & [T] & 0 & 0 \\ 0 & 0 & [T] & 0 \\ 0 & 0 & 0 & [T] \end{bmatrix}$$

$$\cos(x,x') = \frac{x_j - x_i}{L} \qquad \cos(y,x') = \frac{y_j - y_i}{L} \qquad \cos(z,x') = \frac{z_j - z_i}{L}$$

$$D = \sqrt{\cos^2(x,x') + \cos^2(y,x')}$$

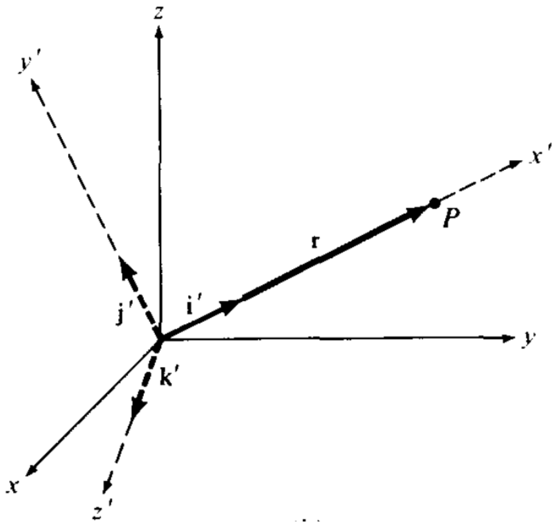
$$\cos(x,y') = \frac{\cos(y,x')}{D} \qquad \cos(y,y') = -\frac{\cos(x,x')}{D} \qquad \cos(z,y') = 0$$

$$\cos(x,z') = -\frac{\cos(x,x')\cos(z,x')}{D} \qquad \cos(y,z') = -\frac{\cos(y,x')\cos(z,x')}{D} \qquad \cos(z,z') = D$$

3D Frames

Transformation Matrix

$$e'_i = T_{ij}\, e_j \quad \Rightarrow \quad \begin{Bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} e'_1 . e_1 & e'_1 . e_2 & e'_1 . e_3 \\ e'_2 . e_1 & e'_2 . e_2 & e'_2 . e_3 \\ e'_3 . e_1 & e'_3 . e_2 & e'_3 . e_3 \end{bmatrix}}_{[T]} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix}$$



$$[R] = \begin{bmatrix} [T] & [0] & [0] \\ [0] & [T] & [0] \\ [0] & [0] & [T] \end{bmatrix}$$

3D Truss Problem

Transformation Matrix

Element stiffness matrix in the global coordinate system

Matrix Form

$$[k] = [R]^T [k^{/}][R]$$

Index Form

$$\{k\} = k_{ij} e_i e_j$$

$$\{k'\} = k'_{mn} e'_m e'_n$$

$e'_m = r_{mi} e_i$

$e'_n = r_{nj} e_j$

$$k_{ij} = k'_{mn}\, r_{mi} r_{nj}$$

3D Frame Problem

Global Stiffness Matrix

$$[k] = [R]^T [k^/][R]$$

Element stiffness matrix in the global coordinate system

Assemblage

3D Frame Problem

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

3D Frame Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$\{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

Once all the displacements are known, the member forces can be easily obtained

$$\{\delta\} \longrightarrow \{\overline{d_3}\} \longrightarrow \{d_3\} = [C_3]^T\{\overline{d_3}\}$$

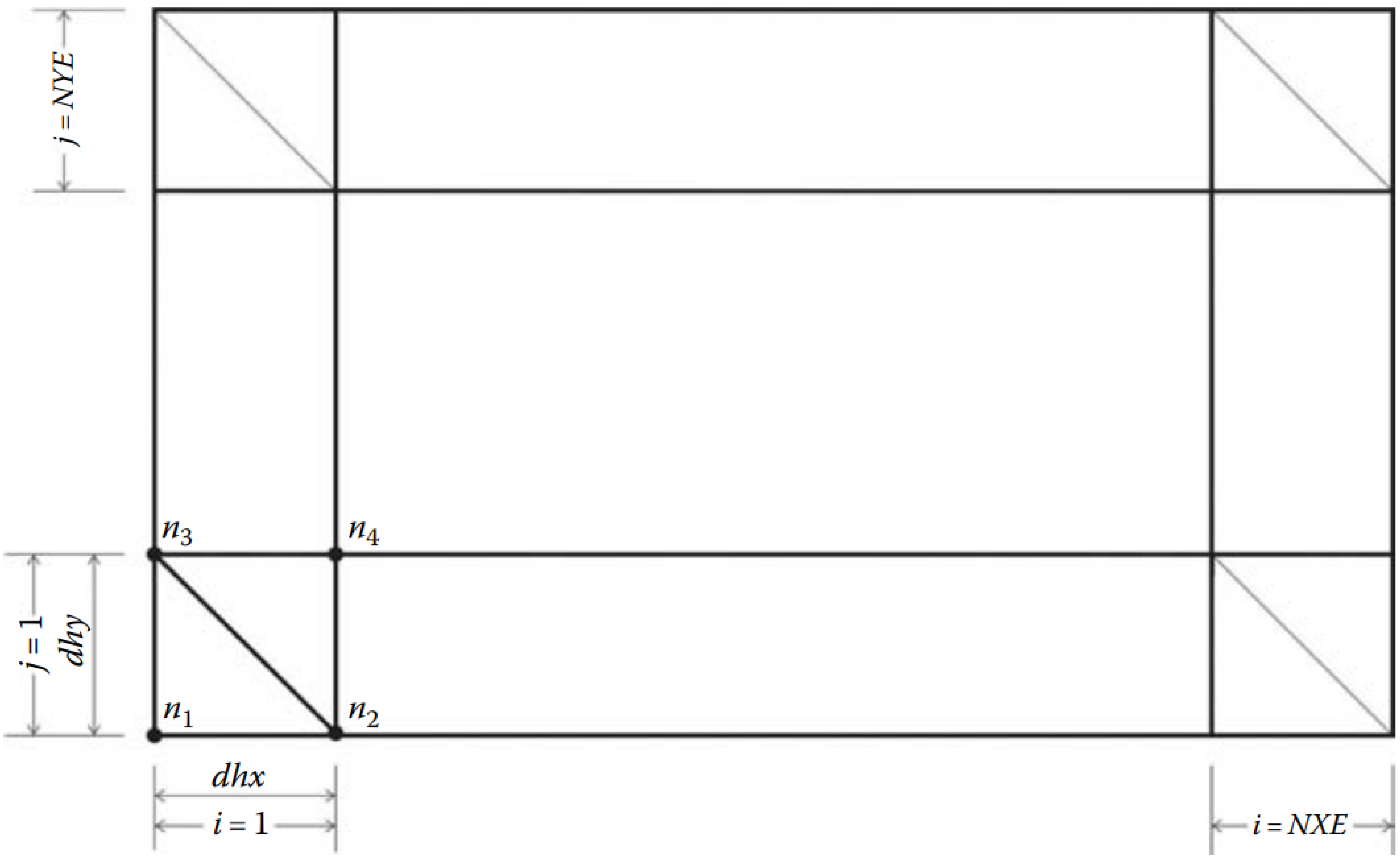
Problem 6: Membrane Problem

Problem Discerption

Problem 6: Membrane Problem

Space Discretization: Mesh Generation

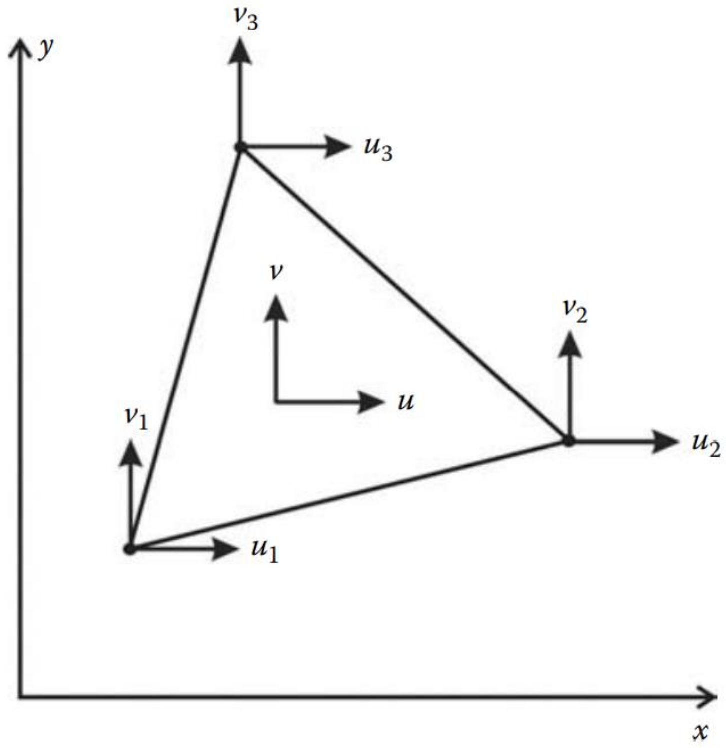
For each interval i and j, four nodes n1, n2, n3, and n4 and **two** elements are created. The first element has nodes n1, n2, n3, while the second element has nodes n2, n4, n3.



$$nel = 2 \times NXE \times NYE$$
$$nnd = (NXE + 1) \times (NYE + 1)$$

Problem 6: Membrane Problem

Interpolation (Shape) Function



$$N_1(x, y) = m_{11} + m_{12}x + m_{13}y$$

$$N_2(x, y) = m_{21} + m_{22}x + m_{23}y$$

$$N_3(x, y) = m_{31} + m_{32}x + m_{33}y$$

$$m_{11} = \frac{x_2 y_3 - x_3 y_2}{2A} \qquad m_{12} = \frac{y_2 - y_3}{2A} \qquad m_{13} = \frac{x_3 - x_2}{2A}$$

$$m_{21} = \frac{x_3 y_1 - x_1 y_3}{2A} \qquad m_{22} = \frac{y_3 - y_1}{2A} \qquad m_{23} = \frac{x_1 - x_3}{2A}$$

$$m_{31} = \frac{x_1 y_2 - x_2 y_1}{2A} \qquad m_{32} = \frac{y_1 - y_2}{2A} \qquad m_{33} = \frac{x_2 - x_1}{2A}$$

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

Problem 6: Membrane Problem

Element Stiffness Matrix: Variational Approach

$$U = \frac{1}{2}\iint_A P\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dA$$

$$T = \frac{1}{2}\iint_A \rho\left(\frac{\partial w}{\partial t}\right)^2 dA$$

$$W = \iint_A f(x,y,t)w(x,y,t)dA$$

$$\delta I = \delta\int_{t_1}^{t_2}\left[\iint_A (U - W - T)\,dA\right] dt = 0$$

$$\delta I = \int_{t_1}^{t_2}\left[\iint_A P\left[\frac{\partial w}{\partial x}\delta\left(\frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial y}\delta\left(\frac{\partial w}{\partial y}\right)\right] dA - \iint_A f(x,y,t)\delta w(x,y,t)dA - \iint_A \rho\frac{\partial^2 w}{\partial t^2}\delta w(x,y,t)\,dA\right] dt = 0$$

$$w = [N]\{a\}$$

$$\delta I = \left(\left(\iint_A P\left[\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right] dA\right)\{a\} - \left(\iint_A [N]^T f(x,y,t)dA\right) - \left(\iint_A [N]^T\rho\,dA\right)\{\ddot{a}\}\right)\delta\{a\} = 0$$

Problem 6: Membrane Problem

Element Stiffness Matrix: Variational Approach

$$\left(\iint_A P\left[\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right]dA\right)\{a\}-\left(\iint_A [N]^T f(x,y,t)dA\right)-\left(\iint_A [N]^T\rho\, dA\right)\{\ddot{a}\} = 0$$

$$[M]\{\ddot{a}(t)\}+[K]\{a(t)\}=F(t)$$

$$[M]=\iint_A [N]^T\rho[N]dA \qquad [K]=\iint_A P\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)dA \qquad F(t)=\iint_A [N]^T f(x,y,t)dA$$

Problem 6: Membrane Problem

Element Stiffness Matrix: Galerkin Approach

$$P\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+f(x,y,t)=\rho\frac{\partial^2 w}{\partial t^2}$$

$w=[N]\{a\}$ ⇒

$$\iint_A [N]^T\left[P\left(\frac{\partial^2 [N]}{\partial x^2}+\frac{\partial^2 [N]}{\partial y^2}\right)\{a\}+f(x,y,t)-\rho[N]\{\ddot{a}\}\right]dA=0$$

$$\iint_A \left[-P\left(\frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x}+\frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right)\{a\}+[N]^T f(x,y,t)-[N]^T\rho[N]\{\ddot{a}\}\right]dA+\oint_C [N]^T P\left(\frac{\partial [N]}{\partial x}n_x+\frac{\partial [N]}{\partial y}n_y\right)dC=0$$

⬇

$$[M]\{\ddot{a}(t)\}+[K]\{a(t)\}=F(t)$$

⬇

$$[M]=\iint_A [N]^T\rho[N]dA \qquad [K]=\iint_A P\left(\frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x}+\frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right)dA \qquad F(t)=\iint_A [N]^T f(x,y,t)dA$$

Problem 6: Membrane Problem

Element Stiffness Matrix

$$[M]=\iint_A [N]^T\rho[N]dA=\iint_{A^e}\begin{bmatrix}L_i\\L_j\\L_k\end{bmatrix}\rho\,[L_i \quad L_j \quad L_k]\,dxdy=\rho\iint_{A^e}\begin{bmatrix}L_i^2 & L_iL_j & L_iL_k\\ L_jL_i & L_j^2 & L_jL_k\\ L_kL_i & L_kL_j & L_k^2\end{bmatrix}=\frac{\rho}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

$$[K]=\iint_A T\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)dA=\iint_{A^e}\begin{bmatrix}m_{21}\\m_{22}\\m_{23}\end{bmatrix}T[m_{21} \quad m_{22} \quad m_{23}]\,dxdy+\iint_{A^e}\begin{bmatrix}m_{31}\\m_{32}\\m_{33}\end{bmatrix}T[m_{31} \quad m_{32} \quad m_{33}]\,dxdy$$

$$=TA\begin{bmatrix}m_{21}^2 & m_{21}m_{22} & m_{21}m_{23}\\ m_{22}m_{21} & m_{22}^2 & m_{22}m_{23}\\ m_{23}m_{21} & m_{23}m_{22} & m_{23}^2\end{bmatrix}+TA\begin{bmatrix}m_{31}^2 & m_{31}m_{32} & m_{31}m_{33}\\ m_{32}m_{31} & m_{32}^2 & m_{32}m_{33}\\ m_{33}m_{31} & m_{33}m_{32} & m_{33}^2\end{bmatrix}$$

$$\{F(t)\}=\iint_A [N]^T P\,dA$$

Problem 6: Membrane Problem

Assemblage

Problem 6: Membrane Problem

Apply Boundary Conditions

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Problem 6: Membrane Problem

Solve (free) Nodal Displacement

$$\{\delta_F\} = [K_{FF}]^{-1} \{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1} \{F_F\}$$

Problem 6: Membrane Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\} \xrightarrow{\text{If } \{\delta_p\} = 0} \{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

To obtain the element stresses and strains, a loop is carried over all the elements:

1. Form element strain matrix **bee** and "steering" vector **g**

a. Loop over the degrees of freedom of the element to obtain element displacements vector **edg**

b. If **g(j) = 0**, then the degree of freedom is restrained; **edg(j) = 0**

c. Otherwise **edg(j) = delta(g(j))**

2. Obtain element strain vector **eps = bee** × **edg**

3. Obtain element stress vector **sigma = dee** × **bee** × **edg**

4. Store the strains for all the elements **EPS(i, :) = eps** for printing to result file

5. Store the stresses for all the elements **SIGMA(i, :) = sigma** for printing to result file

Problem 7: Membrane Problem

Problem Discerption

Problem 7: Membrane Problem

Space Discretization: Mesh Generation

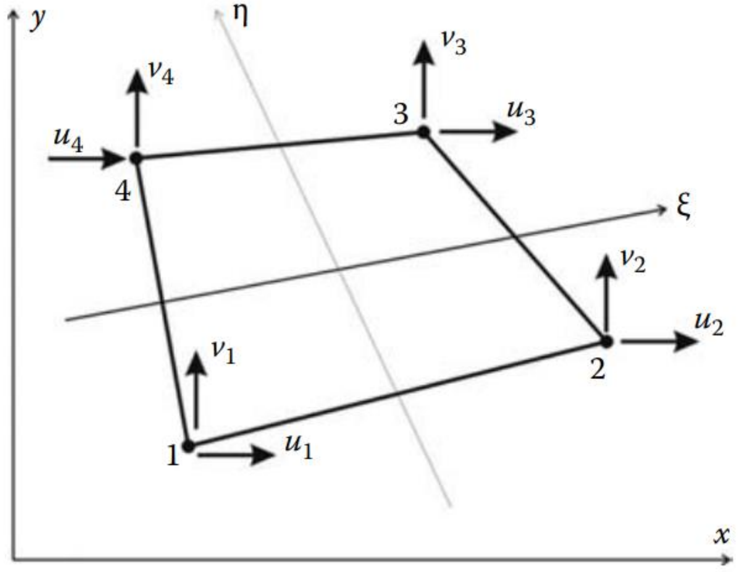
Problem 7: Membrane Problem

Interpolation (Shape) Function

$$w(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi\eta$$

$$N_1(\xi,\eta) = 0.25(1-\xi-\eta+\xi\eta)$$
$$N_2(\xi,\eta) = 0.25(1+\xi-\eta-\xi\eta)$$
$$N_3(\xi,\eta) = 0.25(1+\xi+\eta+\xi\eta)$$
$$N_4(\xi,\eta) = 0.25(1-\xi+\eta-\xi\eta)$$

$$w(\xi,\eta) = N_1w_1 + N_2w_2 + N_3w_3 + N_4w_4$$



$$w(\xi,\eta,t) = [N_1(\xi,\eta) \quad N_2(\xi,\eta) \quad N_3(\xi,\eta) \quad N_4(\xi,\eta)]\begin{Bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{Bmatrix}$$

Problem 7: Membrane Problem

Element Stiffness Matrix: Variational Approach

$$U = \frac{1}{2}\iint_A P\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dA$$

$$T = \frac{1}{2}\iint_A \rho\left(\frac{\partial w}{\partial t}\right)^2 dA$$

$$W = \iint_A f(x,y,t)w(x,y,t)dA$$

$$\delta I = \delta\int_{t_1}^{t_2}\left[\iint_A (U - W - T)\,dA\right] dt = 0$$

$$\delta I = \int_{t_1}^{t_2}\left[\iint_A P\left[\frac{\partial w}{\partial x}\delta\left(\frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial y}\delta\left(\frac{\partial w}{\partial y}\right)\right] dA - \iint_A f(x,y,t)\delta w(x,y,t)dA - \iint_A \rho\frac{\partial^2 w}{\partial t^2}\delta w(x,y,t)\,dA\right] dt = 0$$

$w = [N]\{a\}$

$$\delta I = \left(\left(\iint_A P\left[\frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right] dA\right)\{a\} - \left(\iint_A [N]^T f(x,y,t)dA\right) - \left(\iint_A [N]^T\rho\,dA\right)\{\ddot{a}\}\right)\delta\{a\} = 0$$

Problem 7: Membrane Problem

Element Stiffness Matrix: Variational Approach

$$\left(\iint_A P\left[\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right]dA\right)\{a\}-\left(\iint_A [N]^T f(x,y,t)dA\right)-\left(\iint_A [N]^T\rho\, dA\right)\{\ddot{a}\} = 0$$

$$[M]\{\ddot{a}(t)\}+[K]\{a(t)\}=F(t)$$

$$[M]=\iint_A [N]^T\rho[N]dA \qquad [K]=\iint_A P\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)dA \qquad F(t)=\iint_A [N]^T f(x,y,t)dA$$

Problem 7: Membrane Problem

Element Stiffness Matrix: Galerkin Approach

$$P\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+f(x,y,t)=\rho\frac{\partial^2 w}{\partial t^2} \xrightarrow{w=[N]\{a\}} \iint_A [N]^T\left[P\left(\frac{\partial^2[N]}{\partial x^2}+\frac{\partial^2[N]}{\partial y^2}\right)\{a\}+f(x,y,t)-\rho[N]\{\ddot{a}\}\right]dA=0$$

$$\iint_A \left[-P\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)\{a\}+[N]^T f(x,y,t)-[N]^T\rho[N]\{\ddot{a}\}\right]dA+\oint_C [N]^T P\left(\frac{\partial[N]}{\partial x}n_x+\frac{\partial[N]}{\partial y}n_y\right)dC=0$$

⬇

$$[M]\{\ddot{a}(t)\}+[K]\{a(t)\}=F(t)$$

⬇

$$[M]=\iint_A [N]^T\rho[N]dA \qquad [K]=\iint_A P\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)dA \qquad F(t)=\iint_A [N]^T f(x,y,t)dA$$

Problem 7: Membrane Problem

Element Stiffness Matrix

$$[M] = \iint_A [N]^T \rho [N] dA = \frac{\rho}{9}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 4 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

$$[K] = \iint_A P\left(\frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right) dA$$

$$\{F(t)\} = \iint_A [N]^T f(x,y,t) dA$$

Problem 7: Membrane Problem

Element Stiffness Matrix

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Isoparametric Element →

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

Problem 7: Membrane Problem

Element Stiffness Matrix

$$[M]\{\ddot{a}(t)\} + [K]\{a(t)\} = F(t)$$

$$[M] = \iint_A [N]^T \rho [N] dA = \frac{\rho}{9}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 4 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

$$[K] = \iint_A P\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right) dA$$

$$[K] = P\int_{-1}^{+1}\int_{-1}^{+1}\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)\det[J(\xi,\eta)]d\xi d\eta$$

$$= t\sum_{i=1}^{nhp} W_i[B(\xi_i,\eta_i)]^T[D][B(\xi_i,\eta_i)]\det[J(\xi_i,\eta_i)]$$

Problem 7: Membrane Problem

Assemblage

Problem 7: Membrane Problem

Assemblage

Problem 7: Membrane Problem

Apply Boundary Conditions

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Problem 7: Membrane Problem

Solve (free) Nodal Displacement

$$\{\delta_F\} = [K_{FF}]^{-1} \{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1} \{F_F\}$$

Problem 7: Membrane Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\} \xrightarrow{\text{If } \{\delta_p\} = 0} \{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

To obtain the element stresses and strains, a loop is carried over all the elements:

1. Form element strain matrix **bee** and "steering" vector **g**

a. Loop over the degrees of freedom of the element to obtain element displacements vector **edg**

b. If **g(j) = 0**, then the degree of freedom is restrained; **edg(j) = 0**

c. Otherwise **edg(j) = delta(g(j))**

2. Obtain element strain vector **eps = bee** × **edg**

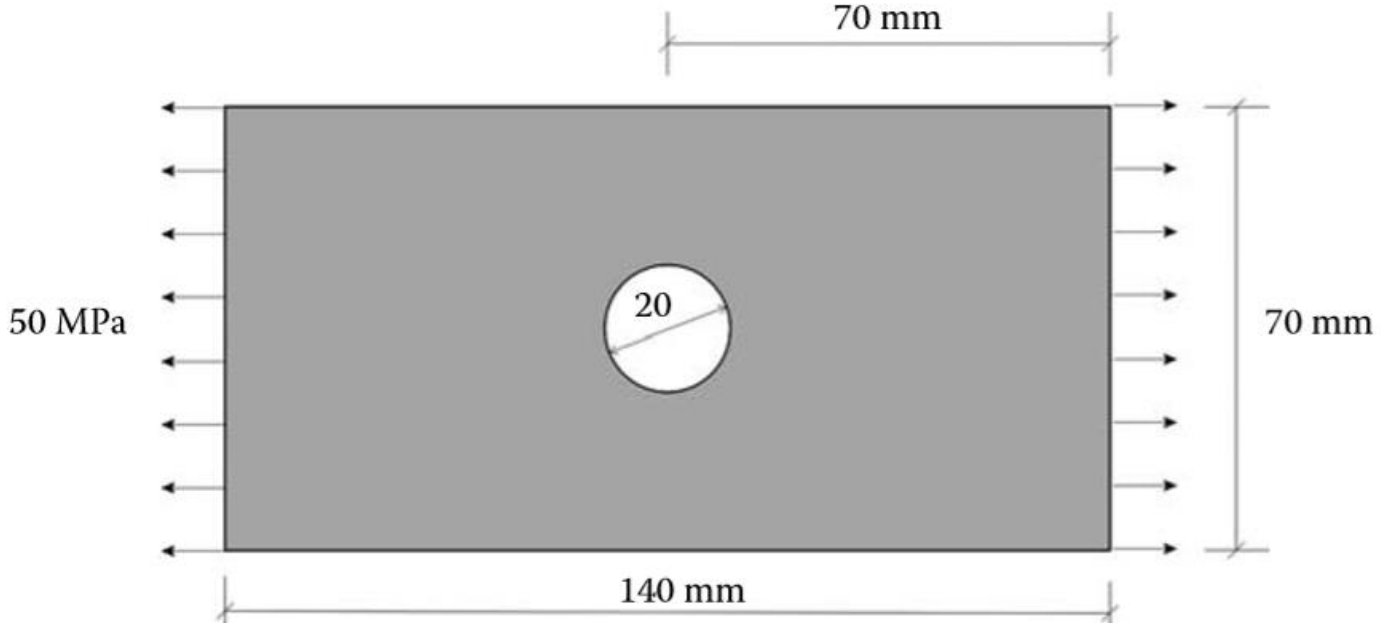
3. Obtain element stress vector **sigma = dee** × **bee** × **edg**

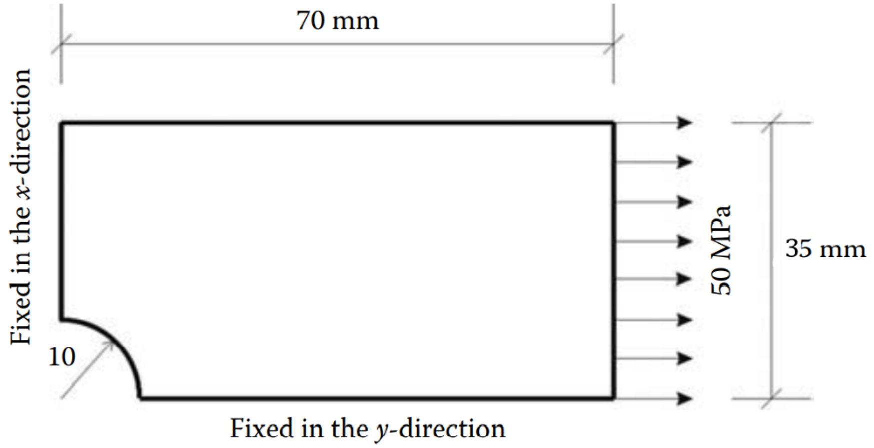
4. Store the strains for all the elements **EPS(i, :) = eps** for printing to result file

5. Store the stresses for all the elements **SIGMA(i, :) = sigma** for printing to result file

Plane Stress Problem: T3

Problem Discerption





$E = 70\ GPa \qquad \nu = 0.33 \qquad Thickness = 2\ mm$

Plane Stress Problem: T3

In reality all solids are three-dimensional. Fortunately, for many practical problems, some simplifying assumptions can be made regarding the stress or strain distributions.

⇩

Such as Plane Stress, Plane Strain, and axisymmetric (symmetry of revolution in both geometry and loading) Problems

Plane stress

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xz} \end{Bmatrix}$$

$$\sigma_{zz} = 0 \quad and \quad \varepsilon_{zz} \neq 0$$

Plane strain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\sigma_{zz} \neq 0 \quad and \quad \varepsilon_{zz} = 0$$

Plane Stress Problem: T3

The infinitesimal strain displacements relations for both theories

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\Rightarrow \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \Rightarrow \{\epsilon\} = [L]U$$

$$u = N_1 u_1 + N_2 u_2 + \cdots + N_n u_n$$
$$v = N_1 v_1 + N_2 v_2 + \cdots + N_n v_n$$

$$\Rightarrow \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & | & N_n & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & | & 0 & N_n \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix} \Rightarrow \{U\} = [N]a$$

Plane Stress Problem: T3

By substitution

$$\begin{cases} \{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}]\{\boldsymbol{U}\} \\ \{\boldsymbol{U}\} = [\boldsymbol{N}]\{\boldsymbol{a}\} \end{cases}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}][\boldsymbol{N}]\{\boldsymbol{a}\} = [\boldsymbol{B}]\{\boldsymbol{a}\}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \dots & | & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & \dots & | & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \dots & | & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Plane Stress Problem: T3

Variational Approach

$$\int_{V_e} \delta\{\epsilon\}^T\{\sigma\}\,dV = \int_{V_e} \delta\{U\}^T\{b\}\,dV + \int_{\Gamma_e} \delta\{U\}^T\{t\}\,d\Gamma + \sum_i \delta\{U\}^T_{(\{x\}=\{\bar{x}\})}\{P\}_i$$

$$\{\delta\epsilon\} = \delta([B]\{a\}) = [B]\{\delta a\} \qquad \{\delta U\} = \delta([N]\{a\}) = [N]\{\delta a\} \qquad \{\sigma\} = [D]\{\epsilon\} = [D][B]\{a\}$$

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$$

$$\left[\int_{A_e} [B]^T[D][B]tdA\right]\{a\} = \int_{A_e} [N]^T\{b\}tdA + \int_{L_e} [N]^T\{t\}tdl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e] = \left[\int_{A_e} [B]^T[D][B]tdA\right] \qquad \{f_e\} = \int_{A_e} [N]^T\{b\}tdA + \int_{L_e} [N]^T\{t\}tdl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e]\{a\} = f_e$$

Plane Stress Problem: T3

Data Preparation (Create Input file)

Nodes Coordinates geom(nnd, 2)

Element Connectivity connec(nel, 3)

Material and Geometrical Properties $E = 70 \times 10^3\ MPa \quad \nu = 0.3$

Boundary Conditions nf(nnd, nodof)

Loading The force in the global force vector **fg**

Plane Stress Problem: T3

Interpolation

Constant Strain Triangle (CST)

$$N_1(x,y) = m_{11} + m_{12}x + m_{13}y$$

$$N_2(x,y) = m_{21} + m_{22}x + m_{23}y$$

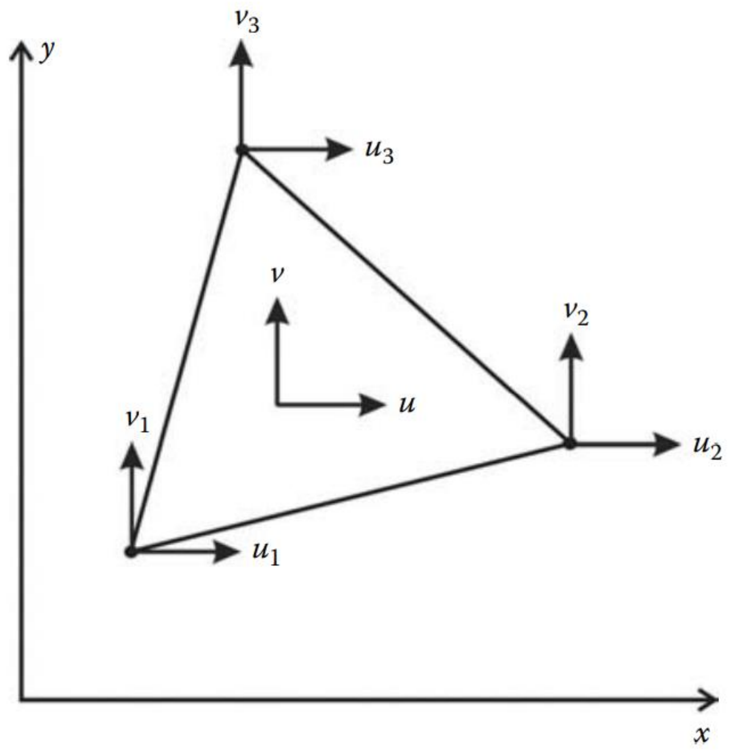
$$N_3(x,y) = m_{31} + m_{32}x + m_{33}y$$

$$m_{11} = \frac{x_2y_3 - x_3y_2}{2A} \qquad m_{12} = \frac{y_2 - y_3}{2A} \qquad m_{13} = \frac{x_3 - x_2}{2A}$$

$$m_{21} = \frac{x_3y_1 - x_1y_3}{2A} \qquad m_{22} = \frac{y_3 - y_1}{2A} \qquad m_{23} = \frac{x_1 - x_3}{2A}$$

$$m_{31} = \frac{x_1y_2 - x_2y_1}{2A} \qquad m_{32} = \frac{y_1 - y_2}{2A} \qquad m_{33} = \frac{x_2 - x_1}{2A}$$

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$



Plane Stress Problem: T3

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & N_3 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \Rightarrow \{U\} = [N]\{a\} \Rightarrow \{\epsilon\} = [B]\{a\}$$

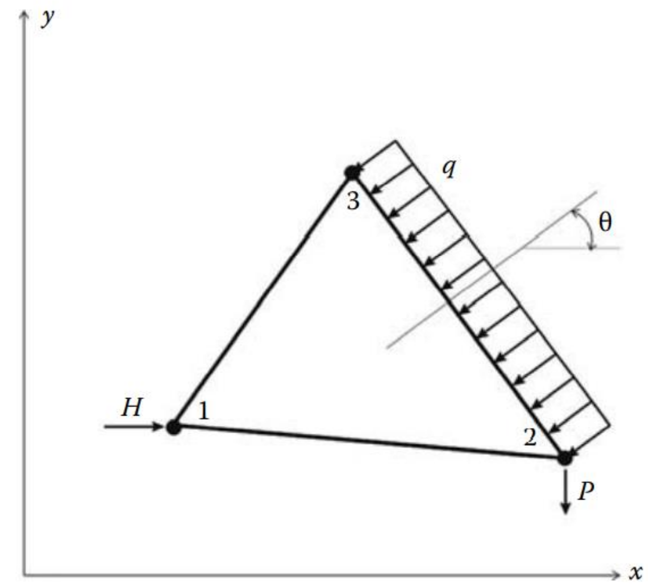
$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \Rightarrow [B] = \begin{bmatrix} m_{12} & 0 & | & m_{22} & 0 & | & m_{32} & 0 \\ 0 & m_{13} & | & 0 & m_{23} & | & 0 & m_{33} \\ m_{13} & m_{12} & | & m_{23} & m_{22} & | & m_{33} & m_{32} \end{bmatrix}$$

Plane Stress Problem: T3

Stiffness Matrix

$$[K_e]\{a\} = f_e$$

$$[K_e] = [B]^T[D][B]tA_e$$



Body Forces

$$\int_{A_e}[N]^T\{b\}t\,dA = -\frac{t}{3}\begin{Bmatrix} 0 \\ \rho g A_e \\ 0 \\ \rho g A_e \\ 0 \\ \rho g A_e \end{Bmatrix}$$

Traction Forces

$$\int_{L_e}[N]^T\{t\}t\,dl = t\begin{Bmatrix} 0 \\ 0 \\ -q\cos\theta L_{2-3}/2 \\ -q\sin\theta L_{2-3}/2 \\ -q\cos\theta L_{2-3}/2 \\ -q\sin\theta L_{2-3}/2 \end{Bmatrix}$$

Concentrated Forces

$$\sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i = \begin{bmatrix} N_1=1 & 0 \\ 0 & N_1=1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} H \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ N_2=1 & 0 \\ 0 & N_2=1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -P \end{Bmatrix} = \begin{Bmatrix} H \\ 0 \\ 0 \\ -P \\ 0 \\ 0 \end{Bmatrix}$$

Plane Stress Problem: T3

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Plane Stress Problem: T3

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\} \xrightarrow{\text{If } \{\delta_p\} = 0} \{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

To obtain the element stresses and strains, a loop is carried over all the elements:

1. Form element strain matrix **bee** and "steering" vector **g**

a. Loop over the degrees of freedom of the element to obtain element displacements vector **edg**

b. If **g(j) = 0**, then the degree of freedom is restrained; **edg(j) = 0**

c. Otherwise **edg(j) = delta(g(j))**

2. Obtain element strain vector **eps = bee** × **edg**

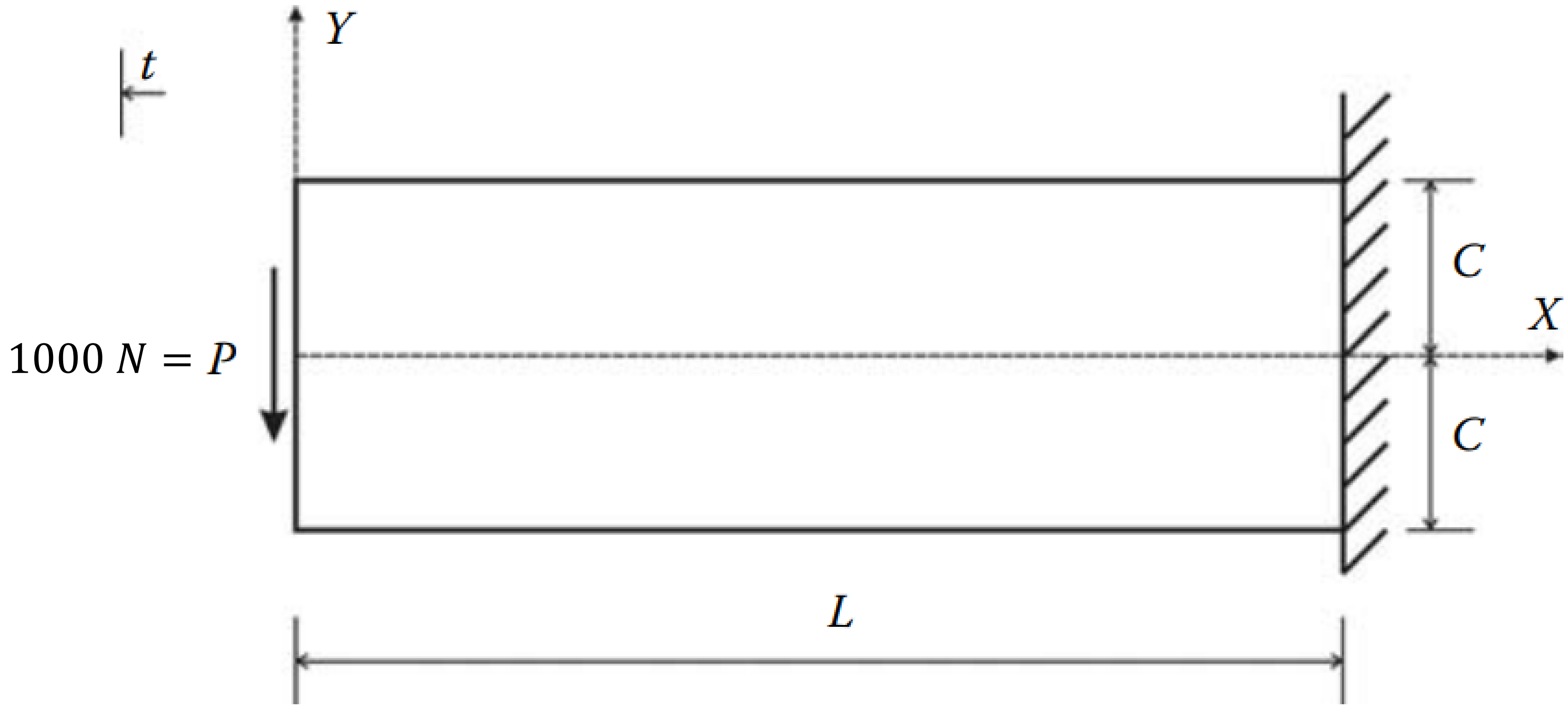
3. Obtain element stress vector **sigma = dee** × **bee** × **edg**

4. Store the strains for all the elements **EPS(i, :) = eps** for printing to result file

5. Store the stresses for all the elements **SIGMA(i, :) = sigma** for printing to result file

Plane Stress Problem: T6

Problem Discerption



$L = 60\,mm$ $C = 10\,mm$ $E = 200\,GPa$ $\nu = 0.3$ $Thickness = 5\,mm$

Plane Stress Problem: T6

In reality all solids are three-dimensional. Fortunately, for many practical problems, some simplifying assumptions can be made regarding the stress or strain distributions.

⇩

Such as Plane Stress, Plane Strain, and axisymmetric (symmetry of revolution in both geometry and loading) Problems

Plane stress

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xz} \end{Bmatrix}$$

$$\sigma_{zz} = 0 \quad and \quad \varepsilon_{zz} \neq 0$$

Plane strain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\sigma_{zz} \neq 0 \quad and \quad \varepsilon_{zz} = 0$$

Plane Stress Problem: T6

The infinitesimal strain displacements relations for both theories

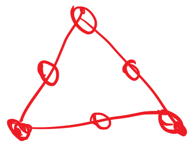
$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\epsilon\} = [L]U$$



$$u = N_1 u_1 + N_2 u_2 + \cdots + N_n u_n$$
$$v = N_1 v_1 + N_2 v_2 + \cdots + N_n v_n$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & | & N_n & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & | & 0 & N_n \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

$$\{U\} = [N]a$$

Plane Stress Problem: T6

By substitution

$$\begin{cases} \{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}]\{\boldsymbol{U}\} \\ \{\boldsymbol{U}\} = [\boldsymbol{N}]\{\boldsymbol{a}\} \end{cases}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}][\boldsymbol{N}]\{\boldsymbol{a}\} = [\boldsymbol{B}]\{\boldsymbol{a}\}$$

مرتب

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \cdots & | & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & \cdots & | & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \cdots & | & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

CST

$\varepsilon = Ba$

$\sigma = D\varepsilon = DBa$

$\begin{bmatrix} m_{22} & 0 & m_{23} & 0 & \cdot \end{bmatrix} = cte$

$\varepsilon_e = B a_e$

Plane Stress Problem: T6

Variational Approach

$$\int_{V_e} \delta\{\epsilon\}^T\{\sigma\}\,dV = \int_{V_e} \delta\{U\}^T\{b\}\,dV + \int_{\Gamma_e} \delta\{U\}^T\{t\}\,d\Gamma + \sum_i \delta\{U\}^T_{(\{x\}=\{\bar{x}\})}\{P\}_i$$

$$\{\delta\epsilon\} = \delta([B]\{a\}) = [B]\{\delta a\} \qquad \{\delta U\} = \delta([N]\{a\}) = [N]\{\delta a\} \qquad \{\sigma\} = [D]\{\epsilon\} = [D][B]\{a\}$$

$$\left[\int_{A_e}[B]^T[D][B]tdA\right]\{a\} = \int_{A_e}[N]^T\{b\}tdA + \int_{L_e}[N]^T\{t\}tdl + \sum_i[N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e] = \left[\int_{A_e}[B]^T[D][B]tdA\right] \qquad \{f_e\} = \int_{A_e}[N]^T\{b\}tdA + \int_{L_e}[N]^T\{t\}tdl + \sum_i[N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e]\{a\} = f_e$$

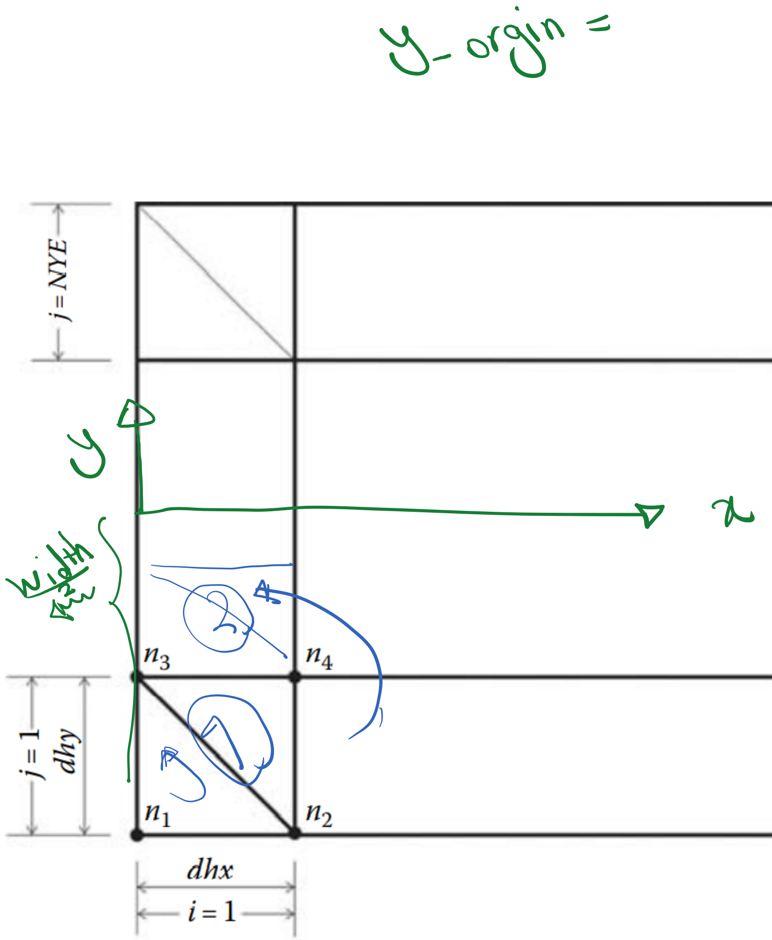
Plane Stress Problem: T6

Data Preparation (Create Input file)

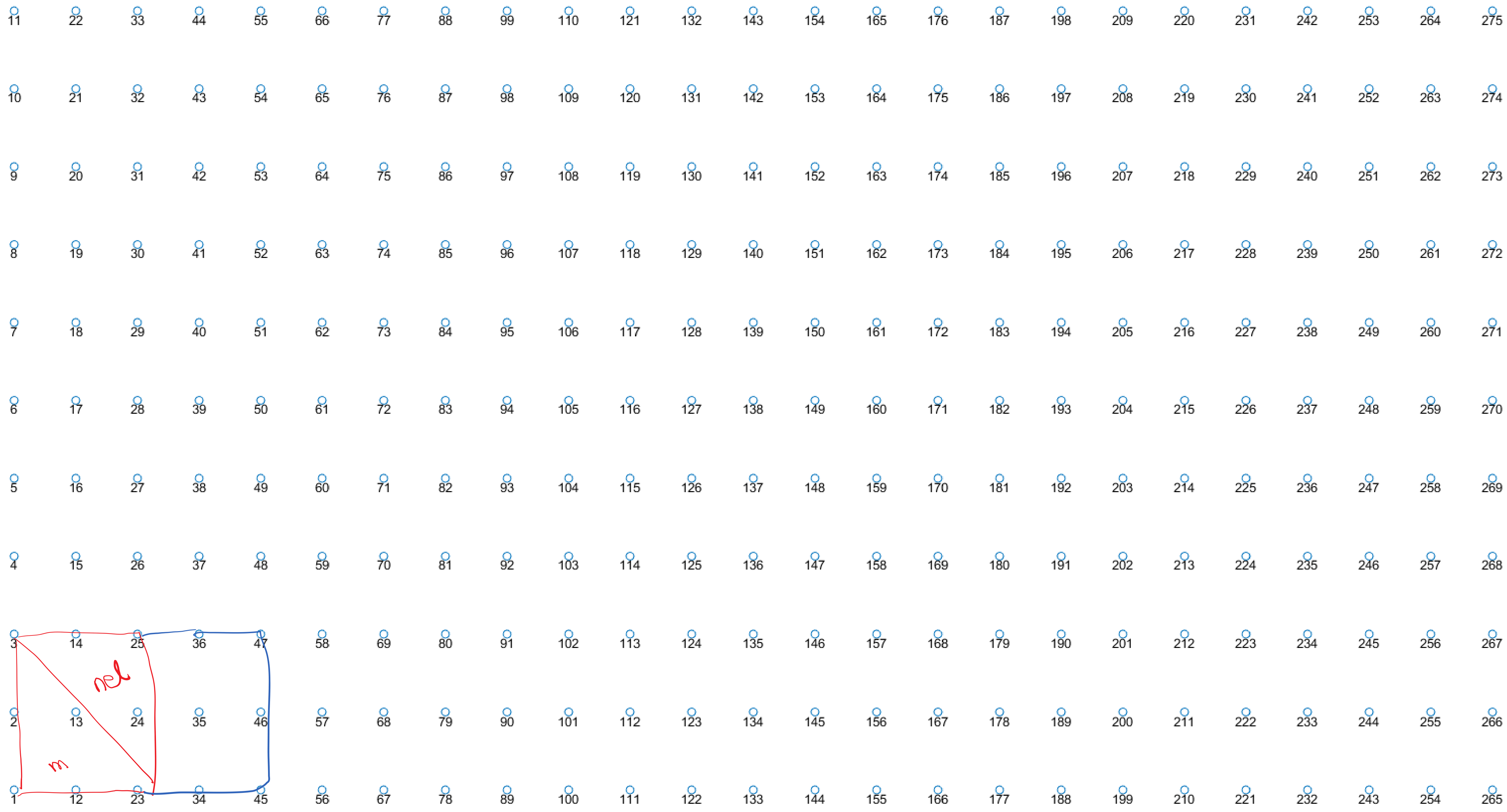
Nodes Coordinates	geom(nnd, 2)
Element Connectivity	connec(nel, nne)
Material and Geometrical Properties	$E = 2 \times 10^5\ MPa \quad \nu = 0.3$
Boundary Conditions	nf(nnd, nodof)
Loading	The force in the global force vector **F**

Plane Stress Problem: T6

Discretization



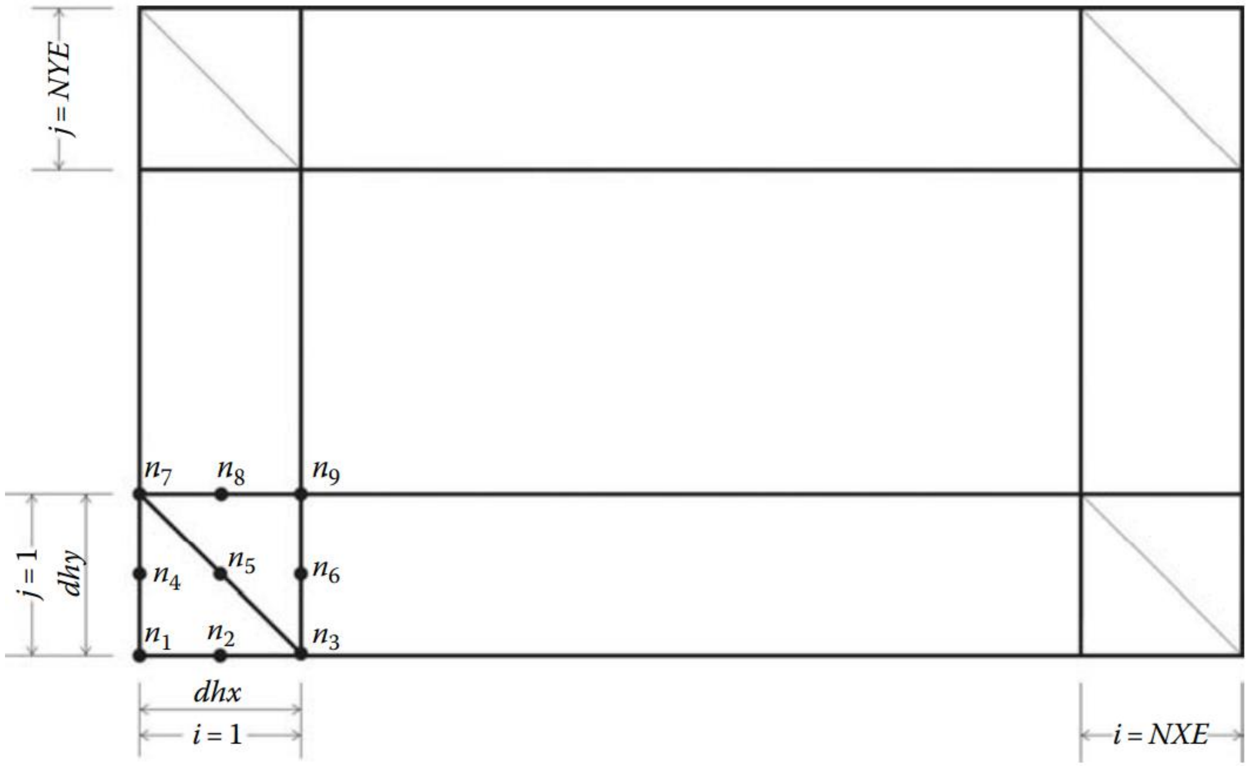

```
nnd = 0;
k = 0;
for i=1:NXE
    for j=1:NYE
        k = k + 1;
        n1 = j + (i-1)*(NYE + 1);
        geom(n1,:) = [(i-1)*dhx-X_origin , (j-1)*dhy-Y_origin ];
        n2 = j + i*(NYE+1);
        geom(n2,:) = [i*dhx-X_origin , (j-1)*dhy-Y_origin ];
        n3 = n1 + 1;
        geom(n3,:) = [(i-1)*dhx-X_origin , j*dhy-Y_origin ];
        n4 = n2 + 1;
        geom(n4,:) = [i*dhx-X_origin , j*dhy-Y_origin ];
        nel = 2*k;
        m = nel -1;
        connec(m,:) = [n1 n2 n3];
        connec(nel,:) = [n2 n4 n3];
        nnd = n4;
    end
end
```


11 22 33 44 55 66 77 88 99 110 121 132 143 154 165 176 187 198 209 220 231 242 253 264 275
10 21 32 43 54 65 76 87 98 109 120 131 142 153 164 175 186 197 208 219 230 241 252 263 274
9 20 31 42 53 64 75 86 97 108 119 130 141 152 163 174 185 196 207 218 229 240 251 262 273
8 19 30 41 52 63 74 85 96 107 118 129 140 151 162 173 184 195 206 217 228 239 250 261 272
7 18 29 40 51 62 73 84 95 106 117 128 139 150 161 172 183 194 205 216 227 238 249 260 271
6 17 28 39 50 61 72 83 94 105 116 127 138 149 160 171 182 193 204 215 226 237 248 259 270
5 16 27 38 49 60 71 82 93 104 115 126 137 148 159 170 181 192 203 214 225 236 247 258 269
4 15 26 37 48 59 70 81 92 103 114 125 136 147 158 169 180 191 202 213 224 235 246 257 268
3 14 25 36 47 58 69 80 91 102 113 124 135 146 157 168 179 190 201 212 223 234 245 256 267
2 13 24 35 46 57 68 79 90 101 112 123 134 145 156 167 178 189 200 211 222 233 244 255 266
1 12 23 34 45 56 67 78 89 100 111 122 133 144 155 166 177 188 199 210 221 232 243 254 265
nel
m

nel = 2K
m = 2K-1

Plane Stress Problem: T6

Discretization




```
nnd = 0;        k = 0;
for  i=1:NXE
        for  j=1:NYE
                k = k + 1;
                n1 = (2*j-1) + (2*i-2)*(2*NYE+1) ;        n2 = (2*j-1) + (2*i-1)*(2*NYE+1);
                n3 = (2*j-1) + (2*i)*(2*NYE+1);
                n4 = n1 + 1;  n5 = n2 + 1;  n6 = n3 + 1;
                n7 = n1 + 2;   n8 = n2 + 2;   n9 = n3 + 2;
                %
                geom(n1,:) = [(i-1)*dhx - X_origin , (j-1)*dhy - Y_origin];
                geom(n2,:) = [((2*i-1)/2)*dhx - X_origin , (j-1)*dhy - Y_origin ];
                geom(n3,:) = [i*dhx - X_origin , (j-1)*dhy - Y_origin ];
                geom(n4,:) = [(i-1)*dhx - X_origin , ((2*j-1)/2)*dhy - Y_origin ];
                geom(n5,:) = [((2*i-1)/2)*dhx - X_origin , ((2*j-1)/2)*dhy - Y_origin ];
                geom(n6,:) = [i*dhx - X_origin , ((2*j-1)/2)*dhy - Y_origin ];
                geom(n7,:) = [(i-1)*dhx - X_origin , j*dhy - Y_origin];
                geom(n8,:) = [((2*i-1)/2)*dhx - X_origin , j*dhy - Y_origin];
                geom(n9,:) = [i*dhx - X_origin , j*dhy - Y_origin];
                %
                nel = 2*k;
                m = nel -1;
                connec(m,:) = [n1 n2 n3 n5 n7 n4];
                connec(nel,:) = [n3 n6 n9 n8 n7 n5];
                nnd= max([n1 n2 n3 n4 n5 n6 n7 n8 n9]);
                % XIN and YIN are two vectors that holds the coordinates X and Y
                % of the grid necessary for the function contourf (XIN,YIN, stress)
                XIG(2*i-1) = geom(n1,1); XIG(2*i) = geom(n2,1); XIG(2*i+1) = geom(n3,1);
                YIG(2*j-1) = geom(n1,2); YIG(2*j) = geom(n4,2); YIG(2*j+1) = geom(n7,2);
        end
end
```

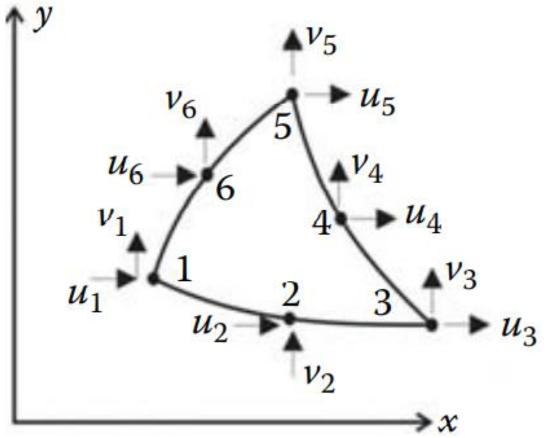
Plane Stress Problem: T6

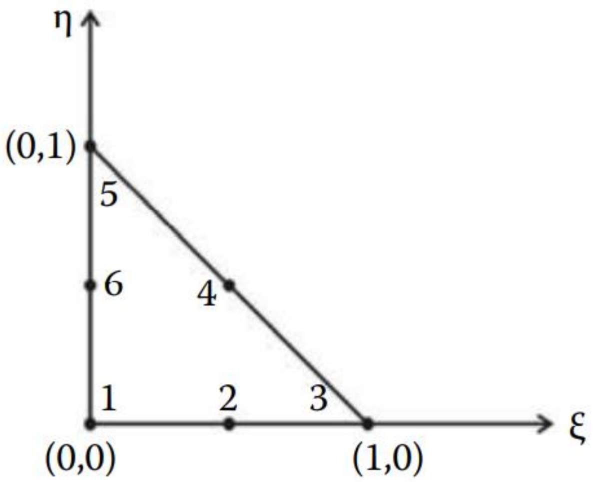
Interpolation

Linear Strain Triangle (LST)

$$\begin{Bmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ N_3(\xi,\eta) \\ N_4(\xi,\eta) \\ N_5(\xi,\eta) \\ N_6(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} -\lambda(1-2\lambda) \\ 4\xi\lambda \\ -\xi(1-2\xi) \\ 4\xi\eta \\ -\eta(1-2\eta) \\ 4\eta\lambda \end{Bmatrix}$$

$$\lambda = 1 - \xi - \eta$$





$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 + N_5v_5 + N_6v_6$$

$$\Rightarrow \quad \{U\} = [N]\{a\}$$

Plane Stress Problem: T6

Stiffness Matrix

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5 + N_6 u_6$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 + N_5 v_5 + N_6 v_6$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}$$

$$\{\epsilon\} = [B]\{a\}$$

$$[B] = \left[\begin{array}{cc|cc|cc|cc|cc|cc} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_5}{\partial x} & 0 & \frac{\partial N_6}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} & 0 & \frac{\partial N_5}{\partial y} & 0 & \frac{\partial N_6}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial y} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial y} & \frac{\partial N_6}{\partial x} \end{array}\right]$$

Plane Stress Problem: T6

Stiffness Matrix

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \Rightarrow [J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{6} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{6} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{6} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{6} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 + N_5x_5 + N_6x_6 = \sum_{i=1}^{6} N_i x_i \rightarrow \frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 + N_5y_5 + N_6y_6 = \sum_{i=1}^{6} N_i y_i$$

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_6}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_6}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_6 & y_6 \end{bmatrix} \Rightarrow [J] = \frac{1}{4}\begin{bmatrix} 1-4\lambda & 4(\lambda-\xi) & -1+4\xi & 4\eta & 0 & -4\eta \\ 1-4\lambda & -4\xi & 0 & 4\xi & -1+4\eta & 4(\lambda-\eta) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix}$$

Der — der — coord

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} \rightarrow \text{der}$$

Plane Stress Problem: T6

Stiffness Matrix

$$[K_e]\{a\} = f_e$$

$$[K_e] = \left[\int_{A_e} [B]^T [D][B] t \, dA \right]$$

$$[K_e] = t \int_0^{+1} \int_0^{1-\xi} [B(\xi, \eta)]^T [D][B(\xi, \eta)] \det[J(\xi, \eta)] d\eta \, d\xi$$

$$= t \sum_{i=1}^{nhp} W_i [B(\xi_i, \eta_i)]^T [D][B(\xi_i, \eta_i)] \det[J(\xi_i, \eta_i)]$$

Plane Stress Problem: T6

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

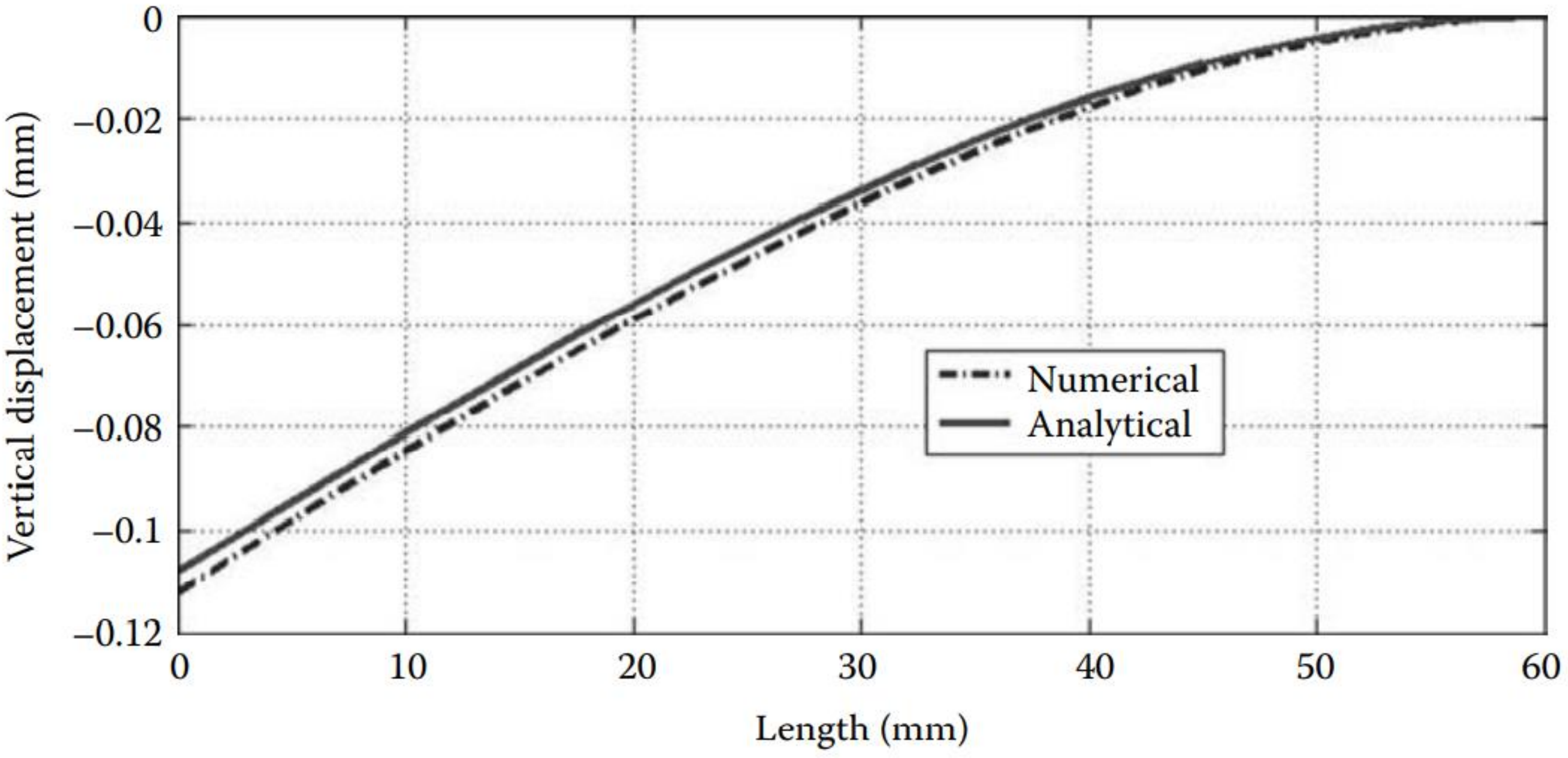
$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Plane Stress Problem: T6

Deflection of the Neutral Line of Cantilever Beam

$$v = \frac{\nu P x y^2}{2EI} + \frac{Px^3}{6EI} - \frac{PL^2 x}{2EI} + \frac{PL^3}{3EI}$$



Plane Stress Problem: T6

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\} \xrightarrow{\text{If } \{\delta_p\} = 0} \{F_P\} = [K_{PF}]\{\delta_F\}$$

MEMBERS' FORCES

To obtain the element stresses and strains, a loop is carried over all the elements:

1. Form element strain matrix **bee** and "steering" vector **g**

a. Loop over the degrees of freedom of the element to obtain element displacements vector **edg**

b. If **g(j) = 0**, then the degree of freedom is restrained; **edg(j) = 0**

c. Otherwise **edg(j) = delta(g(j))**

2. Obtain element strain vector **eps = bee** × **edg**

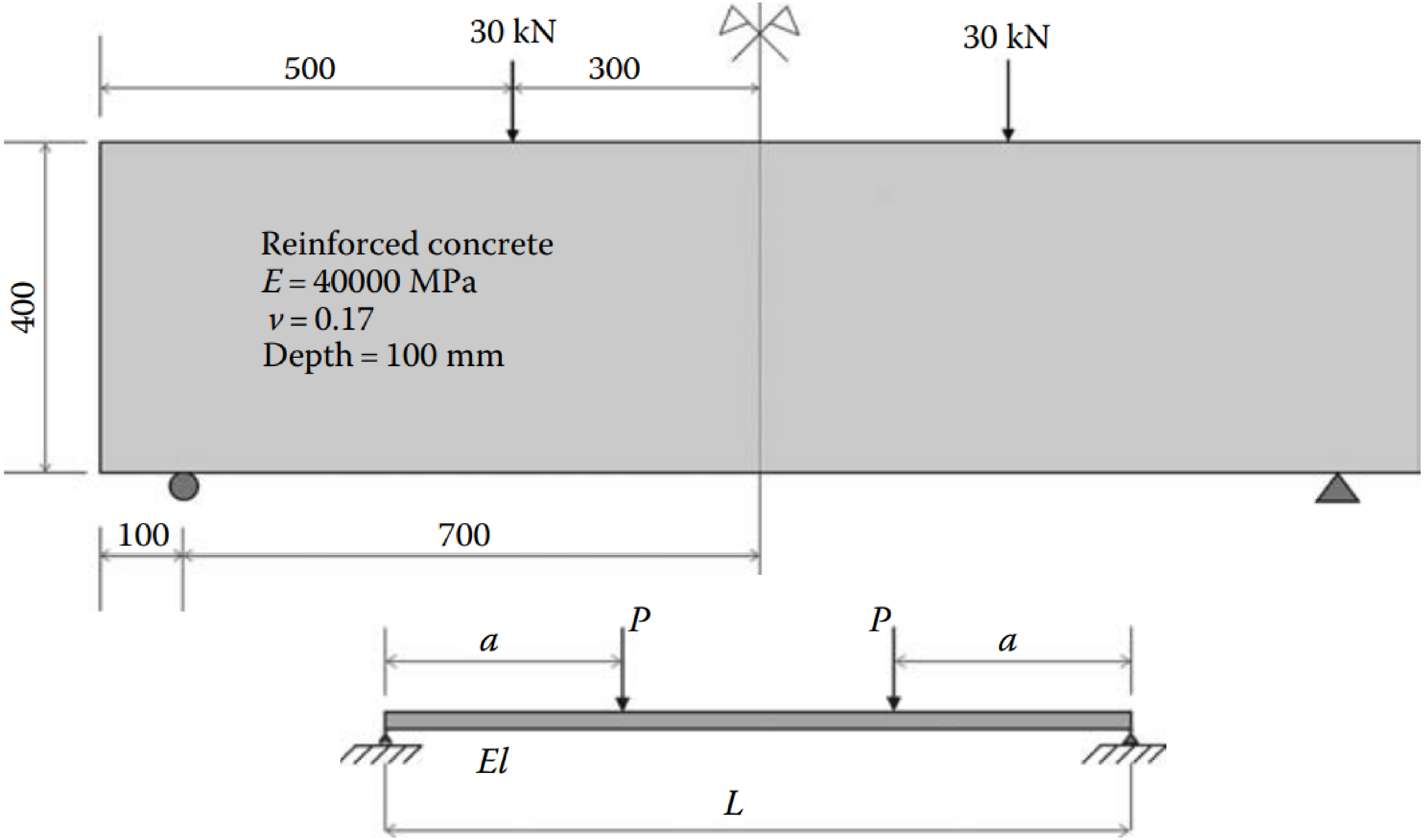
3. Obtain element stress vector **sigma = dee** × **bee** × **edg**

4. Store the strains for all the elements **EPS(i, :) = eps** for printing to result file

5. Store the stresses for all the elements **SIGMA(i, :) = sigma** for printing to result file

Plane Stress Problem: Q4

Problem Discerption



Plane Stress Problem: Q4

In reality all solids are three-dimensional. Fortunately, for many practical problems, some simplifying assumptions can be made regarding the stress or strain distributions.

⇩

Such as Plane Stress, Plane Strain, and axisymmetric (**symmetry of revolution in both geometry and loading**) Problems

Plane stress

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xz} \end{Bmatrix}$$

$$\sigma_{zz} = 0 \quad and \quad \varepsilon_{zz} \neq 0$$

Plane strain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\sigma_{zz} \neq 0 \quad and \quad \varepsilon_{zz} = 0$$

Plane Stress Problem: Q4

The infinitesimal strain displacements relations for both theories

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\Rightarrow \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \Rightarrow \{\epsilon\} = [L]U$$

$$u = N_1 u_1 + N_2 u_2 + \cdots + N_n u_n$$
$$v = N_1 v_1 + N_2 v_2 + \cdots + N_n v_n$$

$$\Rightarrow \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & | & N_n & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & | & 0 & N_n \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix} \Rightarrow \{U\} = [N]a$$

Plane Stress Problem: Q4

By substitution

$$\begin{cases} \{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}]\{\boldsymbol{U}\} \\ \{\boldsymbol{U}\} = [\boldsymbol{N}]\{\boldsymbol{a}\} \end{cases}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}][\boldsymbol{N}]\{\boldsymbol{a}\} = [\boldsymbol{B}]\{\boldsymbol{a}\}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \dots & | & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & \dots & | & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \dots & | & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Plane Stress Problem: Q4

Variational Approach

$$\int_{V_e} \delta\{\epsilon\}^T\{\sigma\}\, dV = \int_{V_e} \delta\{U\}^T\{b\}\, dV + \int_{\Gamma_e} \delta\{U\}^T\{t\}\, d\Gamma + \sum_i \delta\{U\}^T_{(\{x\}=\{\bar{x}\})}\{P\}_i$$

$$\{\delta\epsilon\} = \delta([B]\{a\}) = [B]\{\delta a\}$$

$$\{\delta U\} = \delta([N]\{a\}) = [N]\{\delta a\}$$

$$\{\sigma\} = [D]\{\epsilon\} = [D][B]\{a\}$$

$$\left[\int_{A_e} [B]^T[D][B]tdA\right]\{a\} = \int_{A_e} [N]^T\{b\}tdA + \int_{L_e} [N]^T\{t\}tdl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e] = \left[\int_{A_e} [B]^T[D][B]tdA\right]$$

$$\{f_e\} = \int_{A_e} [N]^T\{b\}tdA + \int_{L_e} [N]^T\{t\}tdl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e]\{a\} = f_e$$

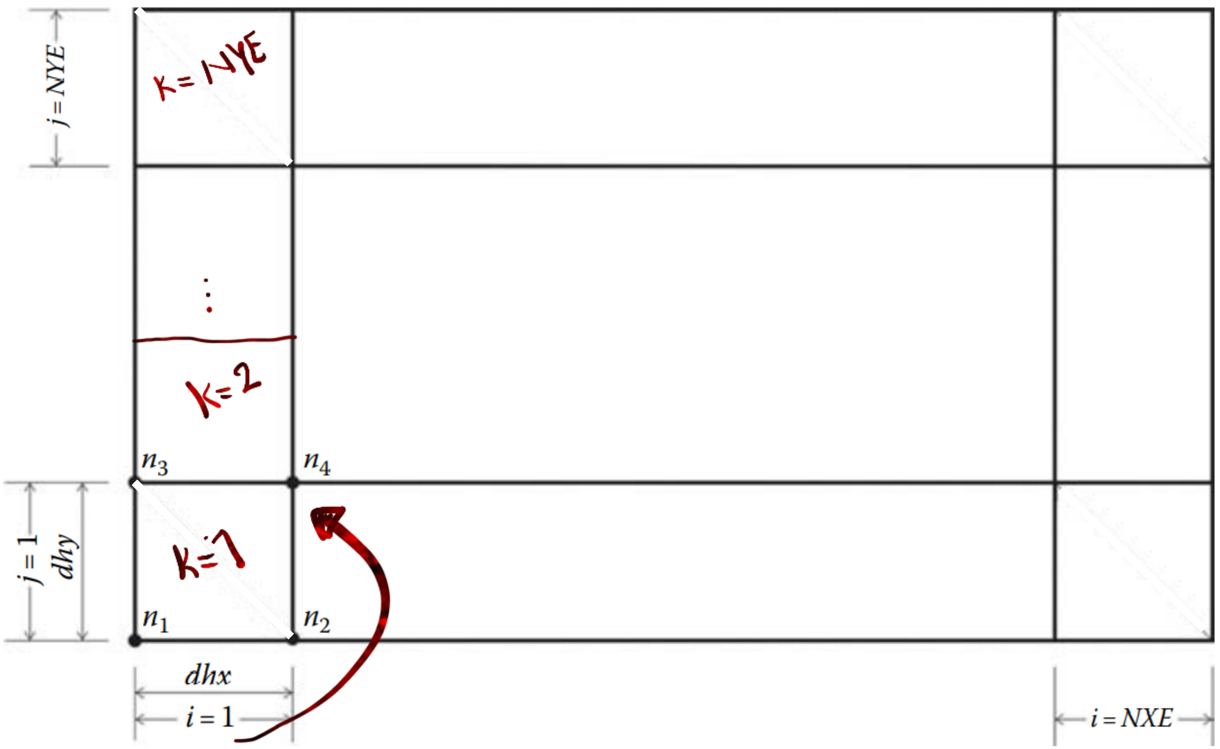
Plane Stress Problem: Q4

Data Preparation (Create Input file)

Nodes Coordinates	geom(nnd, 2)
Element Connectivity	connec(nel, nne)
Material and Geometrical Properties	$E = 4 \times 10^4\ MPa \quad \nu = 0.17$
Boundary Conditions	nf(nnd, nodof)
Loading	The force in the global force vector **fg**

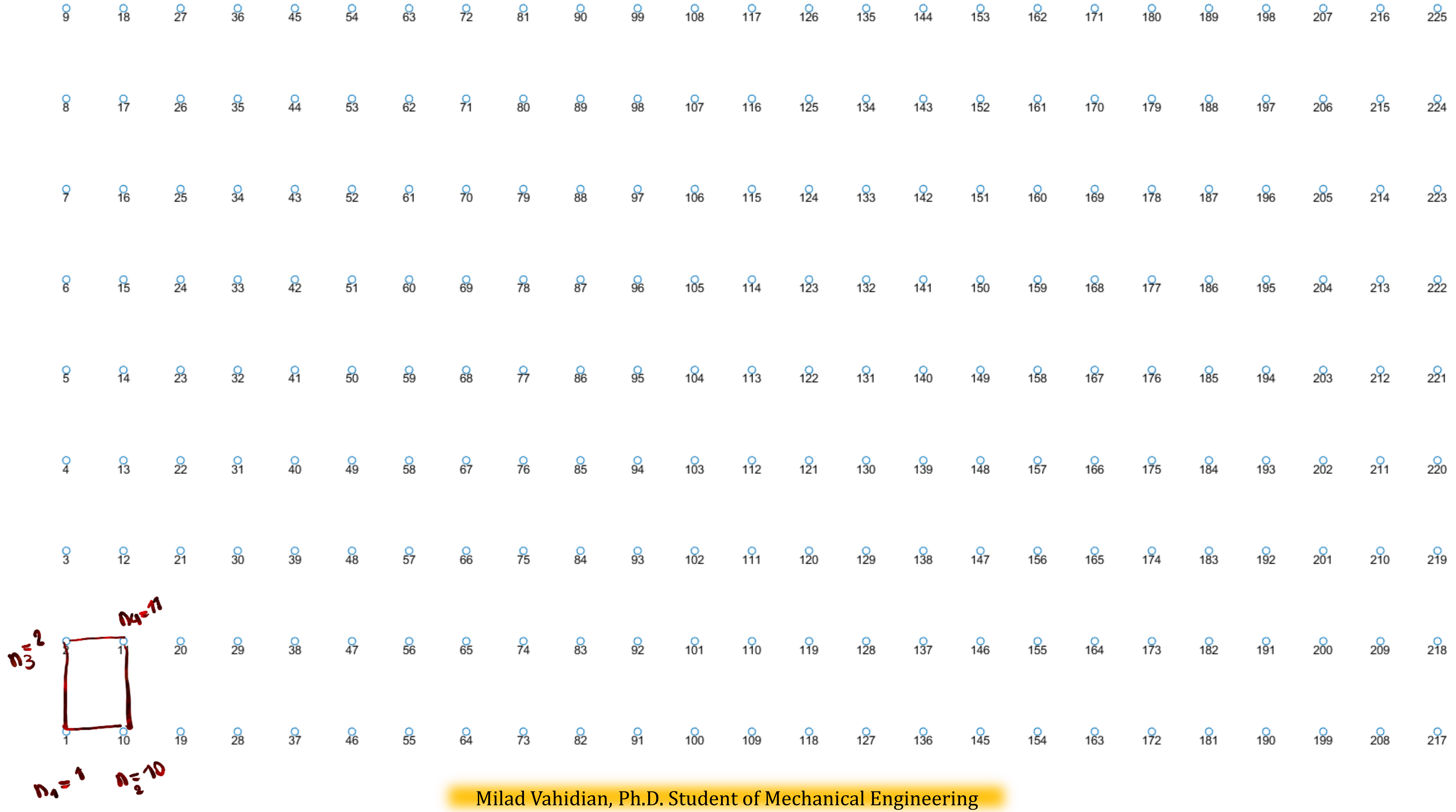
Plane Stress Problem: Q4

Discretization: Mesh Generation




```
nnd =0;
k =0;
for i = 1:NXE
   for j=1:NYE
      k = k + 1;
      n1 = j + (i-1)*(NYE + 1);
      geom(n1,:) = [(i-1)*dhx-X_origin,  (j-1)*dhy-Y_origin ];
      n2 = j + i*(NYE+1);
      geom(n2,:) = [i*dhx-X_origin,  (j-1)*dhy-Y_origin ];
      n3 = n1 + 1;
      geom(n3,:) = [(i-1)*dhx-X_origin,  j*dhy-Y_origin ];
      n4 = n2 + 1;
      geom(n4,:) = [i*dhx-X_origin,  j*dhy-Y_origin ];
      nel = k;
      connec(nel,:) = [n1 n2 n4 n3];
      nnd = n4;
   end
end
```

Plane Stress Problem: Q4

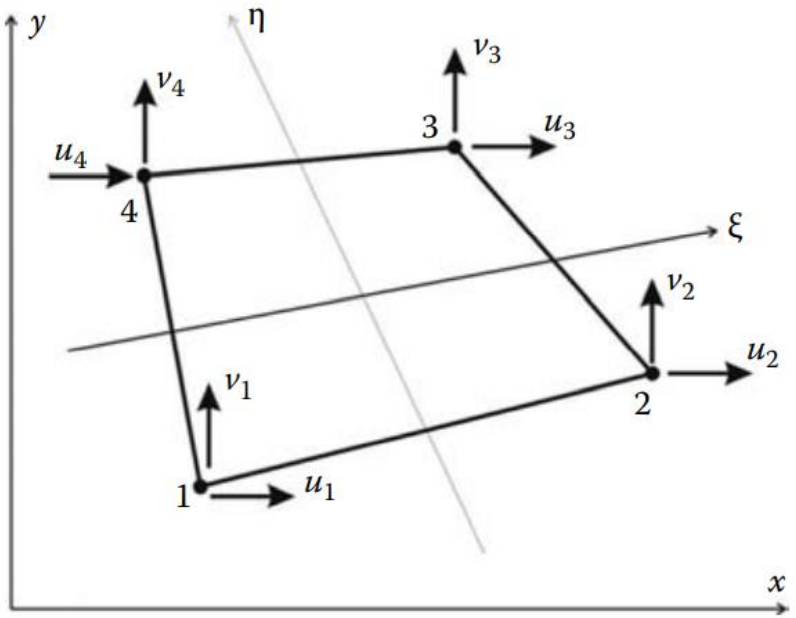


Plane Stress Problem: Q4

Interpolation

Four node Iso-parametric Element

$$N_1(\xi,\eta) = 0.25(1-\xi-\eta+\xi\eta)$$
$$N_2(\xi,\eta) = 0.25(1+\xi-\eta-\xi\eta)$$
$$N_3(\xi,\eta) = 0.25(1+\xi+\eta+\xi\eta)$$
$$N_4(\xi,\eta) = 0.25(1-\xi+\eta-\xi\eta)$$



$u(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi\eta$

$u(-1,-1) = u_1$

⋮

$u(1,1) = u_3$

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$
$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

Plane Stress Problem: Q4

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & \dots & | & N_4 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & \dots & | & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{U\} = [N]\{a\} \quad \xrightarrow{\{\epsilon\} = [L]\{U\}} \quad \{\epsilon\} = [B]\{a\}$$

$$[L][N] = [B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \frac{\partial N_3}{\partial x} & 0 & | & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & 0 & \frac{\partial N_3}{\partial y} & | & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & | & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Plane Stress Problem: Q4

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & N_3 & 0 & | & N_4 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & 0 & N_3 & | & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{U\} = [N]\{a\}$$

$$\{\epsilon\} = [B]\{a\}$$

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

$x = x(\xi, \eta)$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

der   coord

$$[J] = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \cdots & \dfrac{\partial N_4}{\partial \xi} \\ \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \cdots & \dfrac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_4 & y_4 \end{bmatrix}$$

jac

$x = \sum N_i x_i$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Der

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

Plane Stress Problem: Q4

Stiffness Matrix

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_4 & y_4 \end{bmatrix}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$[J] = \underbrace{\frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}}_{\text{der}} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

Plane Stress Problem: Q4

Stiffness Matrix

$$[K_e]\{a\} = f_e$$

$$[K_e] = \left[\int_{A_e} [B]^T [D][B] t \, dA\right]$$

$$\{f_e\} = \int_{A_e} [N]^T \{b\} t \, dA + \int_{L_e} [N]^T \{t\} t \, dl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T \{P\}_i$$

Next Slide

$$[K_e] = t \int_{-1}^{+1} \int_{-1}^{+1} [B(\xi,\eta)]^T [D][B(\xi,\eta)] det[J(\xi,\eta)] d\eta \, d\xi$$

$$= t \sum_{i=1}^{ngp} \sum_{j=1}^{ngp} W_i W_j [B(\xi_i,\eta_j)]^T [D][B(\xi_i,\eta_j)] det[J(\xi_i,\eta_j)]$$

$t = 1 + x + y$

Plane Stress Problem: Q4

Numerical Integration of the Stiffness Matrix

Integration of the Stiffness Matrix for each element is evaluated as follows:

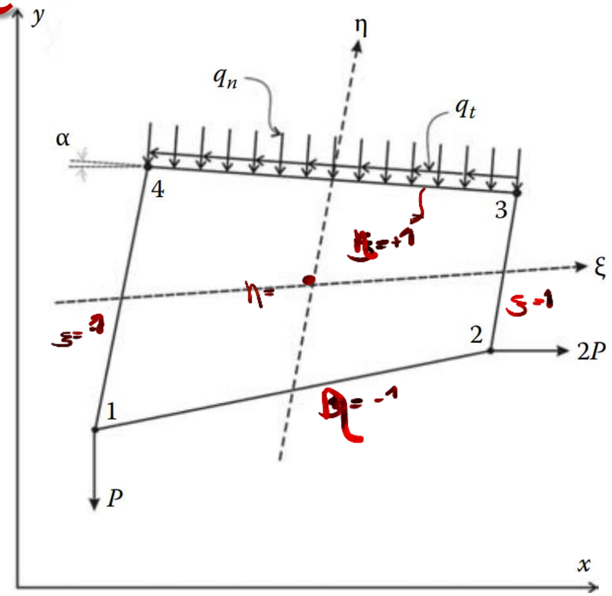
1. For every element i = 1 to nel
2. Retrieve the coordinates of its nodes coord(nne, 2) and its steering vector g(eldof) using the function elem_Q4.m
3. Initialize the stiffness matrix to zero **a.** Loop over the Gauss points ig = 1 to ngp **b.** Retrieve the weight wi as samp(ig, 2)

i. Loop over the Gauss points jg = 1 to ngp
ii. Retrieve the weight wj as samp(jg, 2)
iii. Use the function fmlin.m to compute the shape functions, vector fun, and their derivatives, matrix der, in local coordinates, ξ = samp(ig, 1) and η = samp(jg, 1).
iv. Evaluate the Jacobian jac = der ∗ coord v. Evaluate the determinant of the Jacobian as d = det(jac) vi. Compute the inverse of the Jacobian as jac1 = inv(jac)
vii. Compute the derivatives of the shape functions with respect to the global coordinates x and y as deriv = jac1 ∗ der
viii. Use the function formbee.m to form the strain matrix bee ix. Compute the stiffness matrix as ke = ke + d ∗ thick ∗ wi ∗ wj ∗ B ∗ D ∗ B

4. Assemble the stiffness matrix ke into the global matrix kk

Plane Stress Problem: Q4

Force Vectors



Body Forces

$$\int_{A_e}[N]^T\{b\}t\,dA = t\sum_{i=1}^{ngp}\sum_{j=1}^{ngp} W_iW_j[N(\xi_i,\eta_j)]^T\begin{Bmatrix}0\\-\rho g\end{Bmatrix} det[J(\xi_i,\eta_j)]$$

Traction Forces

$$q_x = q_t dL\cos\alpha - q_n dL\sin\alpha = q_t dx - q_n dy$$

$$q_y = q_n dL\cos\alpha + q_t dL\sin\alpha = q_n dx + q_t dy$$

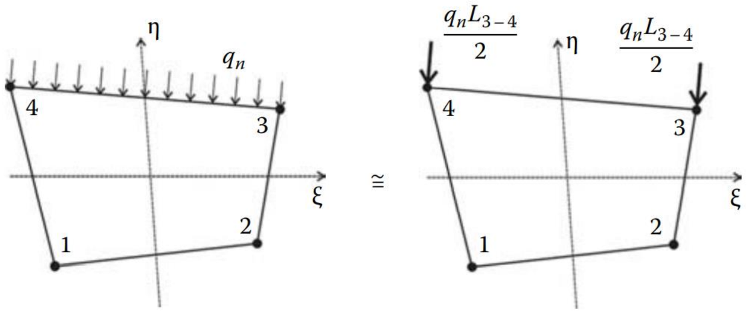
$$q_x = \left(q_t\frac{\partial x}{\partial \xi} - q_n\frac{\partial y}{\partial \xi}\right)d\xi$$

$$q_y = \left(q_n\frac{\partial x}{\partial \xi} + q_t\frac{\partial y}{\partial \xi}\right)d\xi$$

$$\int_{A_e}[N]^T\begin{Bmatrix}q_x\\q_y\end{Bmatrix}dA = t\int_{L_{3-4}}[N(\xi,+1)]^T\begin{Bmatrix}q_x\\q_y\end{Bmatrix}dl$$

$$= t\sum_{i=1}^{ngp} W_i[N(\xi_i,+1)]^T\begin{Bmatrix}\left(q_t\frac{\partial x(\xi_i,+1)}{\partial \xi} - q_n\frac{\partial y(\xi_i,+1)}{\partial \xi}\right)\\ \left(q_n\frac{\partial x(\xi_i,+1)}{\partial \xi} + q_t\frac{\partial y(\xi_i,+1)}{\partial \xi}\right)\end{Bmatrix}$$

When the nodes of an element are numbered anticlockwise a tangential force, such as q_t, is positive if it acts anticlockwise. A normal force, such as q_n, is positive if it acts toward the interior of the element



Concentrated Forces

$$\sum_{k=1}[N]_{x=x_k}\{P_k\} = \begin{bmatrix}1&0\\0&1\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix}\begin{Bmatrix}0\\-P\end{Bmatrix} = \begin{bmatrix}0&0\\0&0\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\end{bmatrix}\begin{Bmatrix}2P\\0\end{Bmatrix} = \begin{Bmatrix}0\\-P\\2P\\0\\0\\0\\0\\0\end{Bmatrix}$$

In practice, when the loads are uniformly distributed they are replaced by equivalent nodal loads. The preceding development is to be used only if the shape of the loading is complicated.

Plane Stress Problem: Q4

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$\{\delta_P\} \to 0$

linsolve(K_{FF}, F_F)

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Plane Stress Problem: Q4

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\} \quad \xrightarrow{\text{If } \{\delta_p\} = 0} \quad \{F_P\} = [K_{PF}]\{\delta_F\}$$

Plane Stress Problem: Q4

Calculation of the Element Resultants

Once the global system of equations is solved, we will compute the stresses at the centroid of the elements. For this we set ngp = 1.

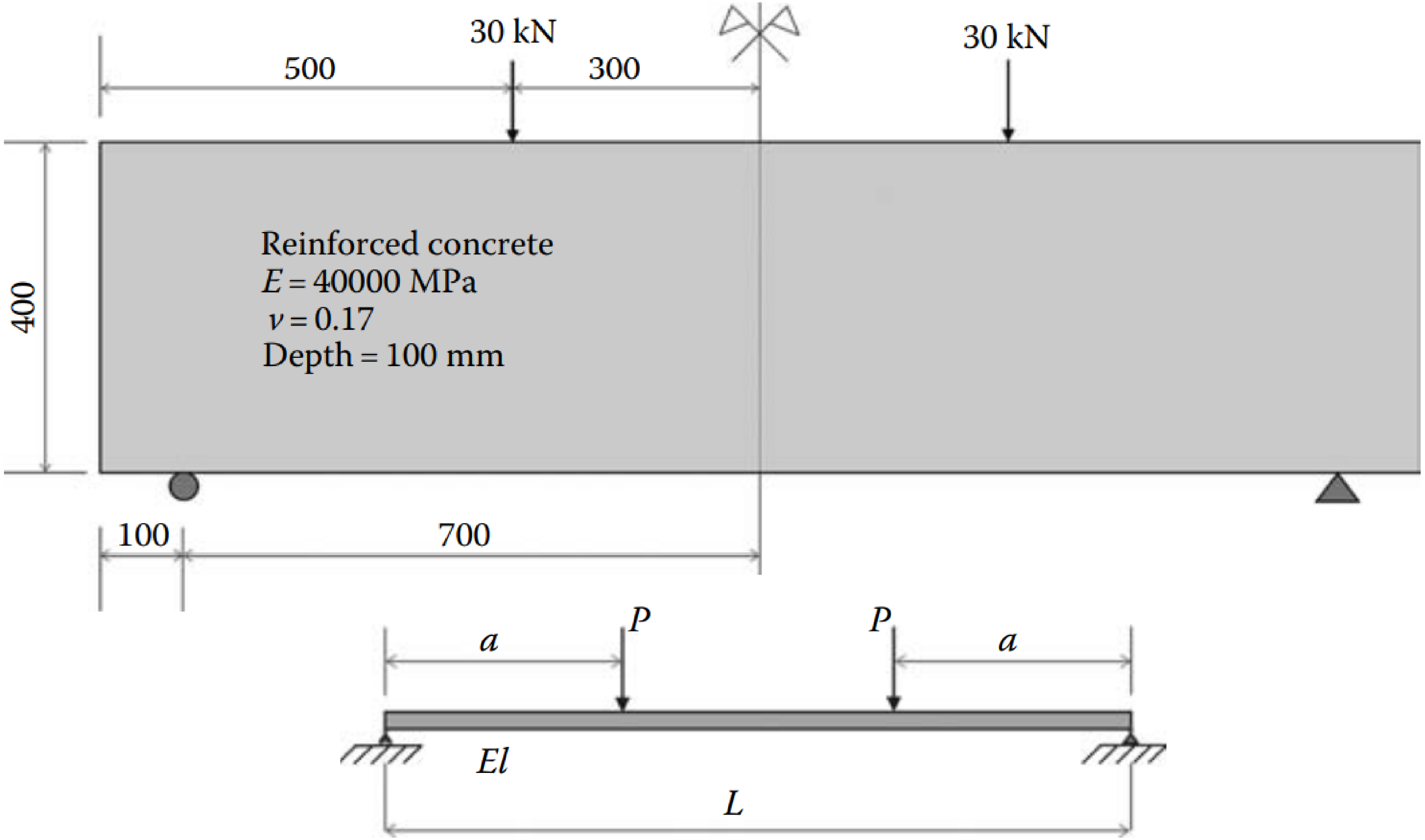
1. For each element
2. Retrieve the coordinates of its nodes coord(nne, 2) and its steering vector g(eldof) using the function elem_Q4.m
3. Retrieve its nodal displacements eld(eldof) from the global vector of displacements delta(n)

a. Loop over the Gauss points ig = 1 to ngp
b. Loop over the Gauss points jg = 1 to ngp
c. Use the function fmlin.m to compute the shape functions, vector fun, and their local derivatives, der, at the local coordinates $\xi = \text{samp}(ig, 1)$ and $\eta = \text{samp}(jg, 1)$
d. Evaluate the Jacobian jac = der ∗ coord
e. Evaluate the determinant of the Jacobian as d = det(jac)
f. Compute the inverse of the Jacobian as jac1 = inv(jac)
g. Compute the derivatives of the shape functions with respect to the global coordinates x and y as deriv = jac1 ∗ der
h. Use the function formbee.m to form the strain matrix bee
i. Compute the strains as eps = bee ∗ eld
j. Compute the stresses as sigma = dee ∗ eps

4. Store the stresses in the matrix SIGMA(nel, 3)

Plane Stress Problem: Q8

Problem Discerption



Plane Stress Problem: Q8

In reality all solids are three-dimensional. Fortunately, for many practical problems, some simplifying assumptions can be made regarding the stress or strain distributions.

Such as Plane Stress, Plane Strain, and axisymmetric (**symmetry of revolution in both geometry and loading**) Problems

Plane stress

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xz} \end{Bmatrix}$$

$$\sigma_{zz} = 0 \quad and \quad \varepsilon_{zz} \neq 0$$

Plane strain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\sigma_{zz} \neq 0 \quad and \quad \varepsilon_{zz} = 0$$

Plane Stress Problem: Q8

The infinitesimal strain displacements relations for both theories

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\Rightarrow \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \Rightarrow \{\epsilon\} = [L]U$$

$$u = N_1 u_1 + N_2 u_2 + \cdots + N_n u_n$$
$$v = N_1 v_1 + N_2 v_2 + \cdots + N_n v_n$$

$$\Rightarrow \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & | & N_n & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & | & 0 & N_n \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix} \Rightarrow \{U\} = [N]a$$

Plane Stress Problem: Q8

By substitution

$$\begin{cases} \{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}]\{\boldsymbol{U}\} \\ \{\boldsymbol{U}\} = [\boldsymbol{N}]\{\boldsymbol{a}\} \end{cases}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}][\boldsymbol{N}]\{\boldsymbol{a}\} = [\boldsymbol{B}]\{\boldsymbol{a}\}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \dots & | & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & \dots & | & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \dots & | & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Plane Stress Problem: Q8

Variational Approach

$$\int_{V_e} \delta\{\epsilon\}^T\{\sigma\}\, dV = \int_{V_e} \delta\{U\}^T\{b\}\, dV + \int_{\Gamma_e} \delta\{U\}^T\{t\}\, d\Gamma + \sum_i \delta\{U\}^T_{(\{x\}=\{\bar{x}\})}\{P\}_i$$

$$\{\delta\epsilon\} = \delta([B]\{a\}) = [B]\{\delta a\}$$

$$\{\delta U\} = \delta([N]\{a\}) = [N]\{\delta a\}$$

$$\{\sigma\} = [D]\{\epsilon\} = [D][B]\{a\}$$

$$\left[\int_{A_e} [B]^T[D][B]tdA\right]\{a\} = \int_{A_e} [N]^T\{b\}tdA + \int_{L_e} [N]^T\{t\}tdl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e] = \left[\int_{A_e} [B]^T[D][B]tdA\right]$$

$$\{f_e\} = \int_{A_e} [N]^T\{b\}tdA + \int_{L_e} [N]^T\{t\}tdl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

$$[K_e]\{a\} = f_e$$

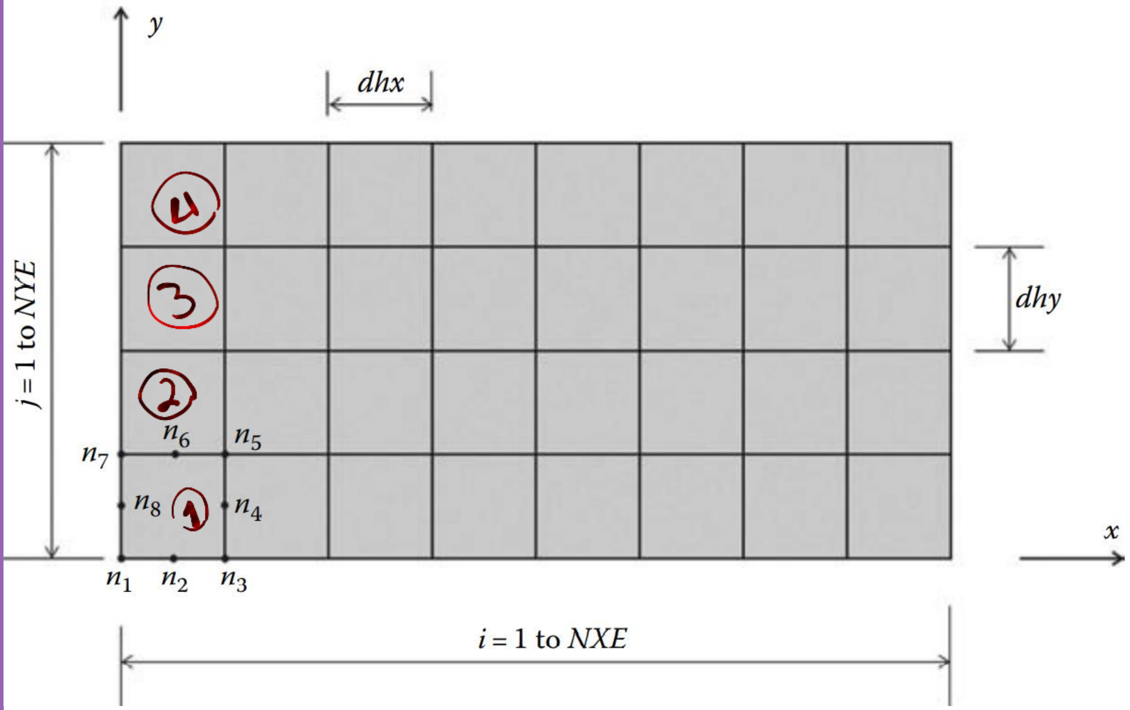
Plane Stress Problem: Q8

Data Preparation (Create Input file)

Nodes Coordinates	geom(nnd, 2)
Element Connectivity	connec(nel, nne)
Material and Geometrical Properties	$E = 4 \times 10^4\ MPa \quad \nu = 0.17$
Boundary Conditions	nf(nnd, nodof)
Loading	The force in the global force vector **fg**

Plane Stress Problem: Q8

Discretization: Mesh Generation

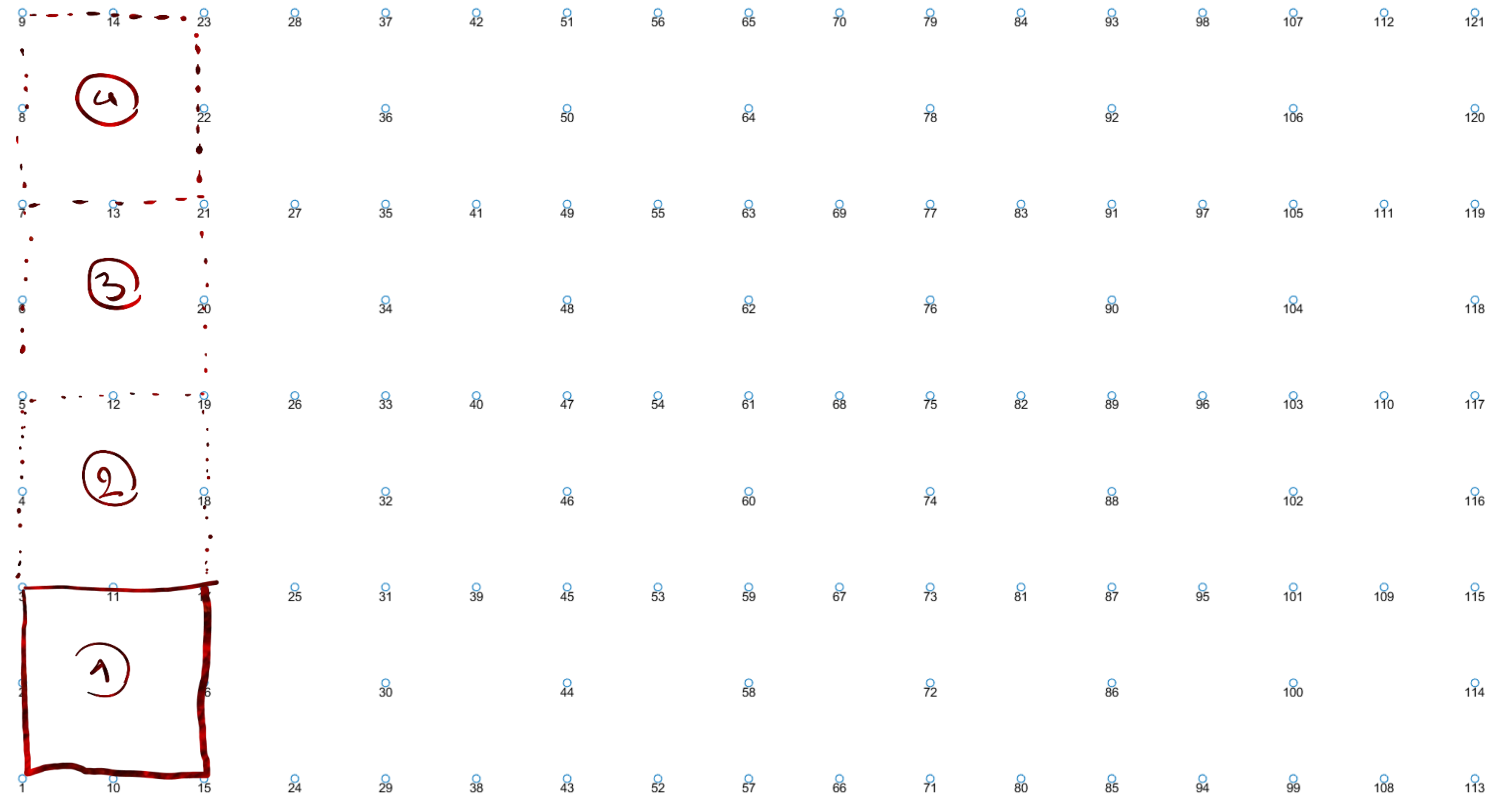



```
nnd=0;
k=0;
for i=1:NXE
   for j=1:NYE
      k=k+1;
      n1=(i-1)*(3*NYE+2)+2*j - 1;
      n2=i*(3*NYE+2)+j - NYE - 1;
      n3=i*(3*NYE+2)+2*j-1;
      n4=n3 + 1;           n5=n3 + 2;           n6=n2 + 1;
      n7=n1 + 2;           n8=n1 + 1;
      geom(n1,:)=[(i-1)*dhx-X_origin,   (j-1)*dhy-Y_origin];
      geom(n3,:)=[i*dhx - X_origin,      (j-1)*dhy-Y_origin];
      geom(n5,:)=[i*dhx-X_origin,         j*dhy - Y_origin];
      geom(n7,:)=[(i-1)*dhx - X_origin,   j*dhy - Y_origin];
      geom(n2,:)=[(geom(n1,1)+geom(n3,1))/2 , (geom(n1,2)+geom(n3,2))/2];
      geom(n4,:)=[(geom(n3,1)+ geom(n5,1))/2 (geom(n3,2)+ geom(n5,2))/2];
      geom(n6,:)=[(geom(n5,1)+ geom(n7,1))/2 (geom(n5,2)+ geom(n7,2))/2];
      geom(n8,:)=[(geom(n1,1)+ geom(n7,1))/2 (geom(n1,2)+ geom(n7,2))/2];
      nel = k;
      nnd = n5;
      connec(k,:)=[n1 n2 n3 n4 n5 n6 n7 n8];
   end
end
```

Plane Stress Problem: Q8

Discretization: Mesh Generation

3NYE-(NYE-1) = 2NY+1
2NYE -(NYE-1) = NYE+1
=(i-1)(3NYE+2)



NYE=4 NXE=8

Plane Stress Problem: Q8

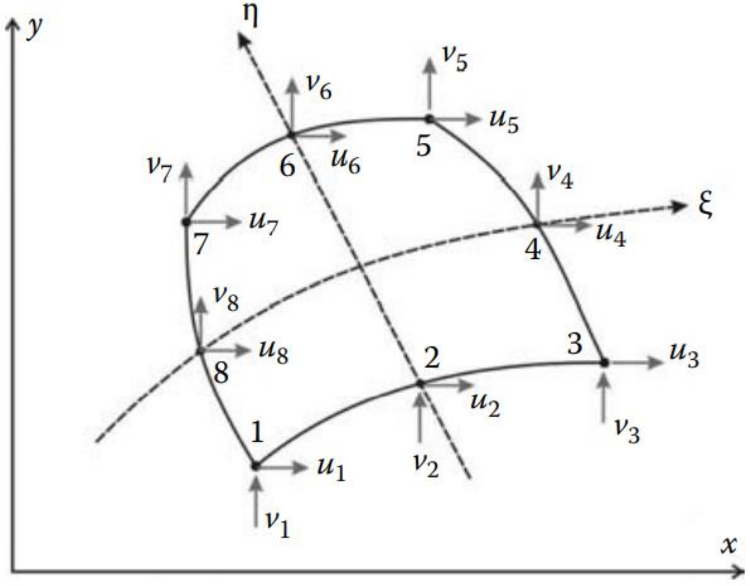
Interpolation

Eight-noded Iso-parametric Element

$$
\begin{Bmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ N_3(\xi,\eta) \\ N_4(\xi,\eta) \\ N_5(\xi,\eta) \\ N_6(\xi,\eta) \\ N_7(\xi,\eta) \\ N_8(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} -0.25(1-\xi)(1-\eta)(1+\xi+\eta) \\ 0.50(1-\xi^2)(1-\eta) \\ -0.25(1+\xi)(1-\eta)(1-\xi+\eta) \\ 0.50(1+\xi)(1-\eta^2) \\ -0.25(1+\xi)(1+\eta)(1-\xi-\eta) \\ 0.50(1-\xi^2)(1+\eta) \\ -0.25(1-\xi)(1+\eta)(1+\xi-\eta) \\ 0.50(1-\xi)(1-\eta^2) \end{Bmatrix}
$$

$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\xi^2 + c_6\eta^2 + c_7\xi\eta^2 + c_8\xi^2\eta$

$u(\xi=-1,\eta=-1) = u_1$



$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6 + N_7u_7 + N_8u_8$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 + N_5v_5 + N_6v_6 + N_7v_7 + N_8v_8$$

Plane Stress Problem: Q8

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & \dots & | & N_8 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & \dots & | & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}$$

$$\{U\} = [N]\{a\} \quad \Rightarrow \quad \{\epsilon\} = [B]\{a\}$$

$N(x,y)$ $\quad \frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \xi}$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

der $\quad$ coord

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \dots & \frac{\partial N_8}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \dots & \frac{\partial N_8}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_8 & y_8 \end{bmatrix}$$

$x = \sum N_i x_i$

$$\begin{cases} x = N_1 x_1 + N_2 x_2 + \dots + N_8 x_8 \\ y = N_1 y_1 + N_2 y_2 + \dots + N_8 y_8 \end{cases}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

Der $\quad$ der

Plane Stress Problem: Q8

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & \dots & | & N_8 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & \dots & | & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix} \Rightarrow \{U\} = [N]\{a\} \Rightarrow \{\epsilon\} = [B]\{a\}$$

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & | & \dfrac{\partial N_2}{\partial x} & 0 & | & \dots & \dots & | & \dfrac{\partial N_8}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & | & 0 & \dfrac{\partial N_2}{\partial y} & | & \dots & \dots & | & 0 & \dfrac{\partial N_8}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & | & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & | & \dots & \dots & | & \dfrac{\partial N_8}{\partial y} & \dfrac{\partial N_8}{\partial x} \end{bmatrix}$$

Plane Stress Problem: Q8

Stiffness Matrix

$$[K_e]\{a\} = f_e$$

$$[K_e] = \left[\int_{A_e} [B]^T [D][B] t \, dA\right]$$

$$\{f_e\} = \int_{A_e} [N]^T \{b\} t \, dA + \int_{L_e} [N]^T \{t\} t \, dl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T \{P\}_i$$

Next Slide

$$[K_e] = t \int_{-1}^{+1} \int_{-1}^{+1} [B(\xi,\eta)]^T [D][B(\xi,\eta)] det[J(\xi,\eta)] d\eta \, d\xi$$

$$= t \sum_{i=1}^{ngp} \sum_{j=1}^{ngp} W_i W_j [B(\xi_i,\eta_j)]^T [D][B(\xi_i,\eta_j)] det[J(\xi_i,\eta_j)]$$

Plane Stress Problem: Q8

Numerical Integration of the Stiffness Matrix

Integration of the Stiffness Matrix for each element is evaluated as follows:

1. For every element i = 1 to nel
2. Retrieve the coordinates of its nodes coord(nne, 2) and its steering vector g(eldof) using the function elem_Q4.m
3. Initialize the stiffness matrix to zero **a.** Loop over the Gauss points ig = 1 to ngp **b.** Retrieve the weight wi as samp(ig, 2)

i. Loop over the Gauss points jg = 1 to ngp
ii. Retrieve the weight wj as samp(jg, 2)
iii. Use the function fmlin.m to compute the shape functions, vector fun, and their derivatives, matrix der, in local coordinates, ξ = samp(ig, 1) and η = samp(jg, 1).
iv. Evaluate the Jacobian jac = der ∗ coord v. Evaluate the determinant of the Jacobian as d = det(jac) vi. Compute the inverse of the Jacobian as jac1 = inv(jac)
vii. Compute the derivatives of the shape functions with respect to the global coordinates x and y as deriv = jac1 ∗ der
viii. Use the function formbee.m to form the strain matrix bee ix. Compute the stiffness matrix as ke = ke + d ∗ thick ∗ wi ∗ wj ∗ B ∗ D ∗ B

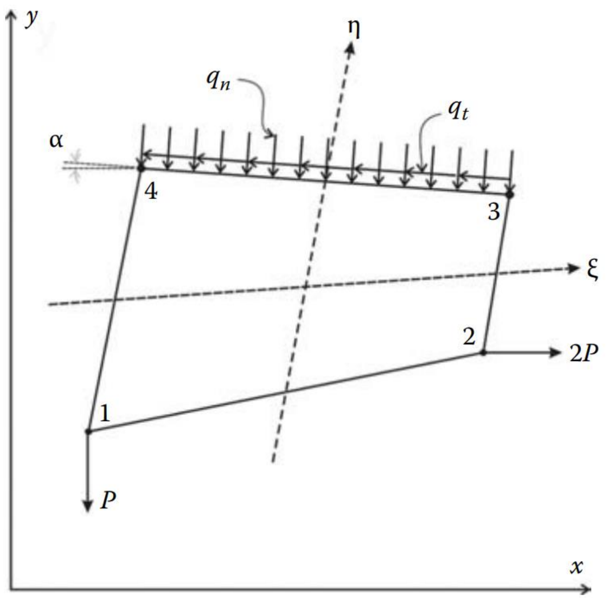
4. Assemble the stiffness matrix ke into the global matrix kk

Plane Stress Problem: Q8

Force Vectors

Body Forces

$$\int_{A_e} [N]^T \{b\} t\, dA = t \sum_{i=1}^{ngp} \sum_{j=1}^{ngp} W_i W_j [N(\xi_i, \eta_j)]^T \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} det[J(\xi_i, \eta_j)]$$



When the nodes of an element are numbered anticlockwise a tangential force, such as q_t, is positive if it acts anticlockwise. A normal force, such as q_n, is positive if it acts toward the interior of the element

Traction Forces

$$q_x = q_t dL \cos\alpha - q_n dL \sin\alpha = q_t dx - q_n dy$$

$$q_y = q_n dL \cos\alpha + q_t dL \sin\alpha = q_n dx + q_t dy$$

$$q_x = \left(q_t \frac{\partial x}{\partial \xi} - q_n \frac{\partial y}{\partial \xi} \right) d\xi$$

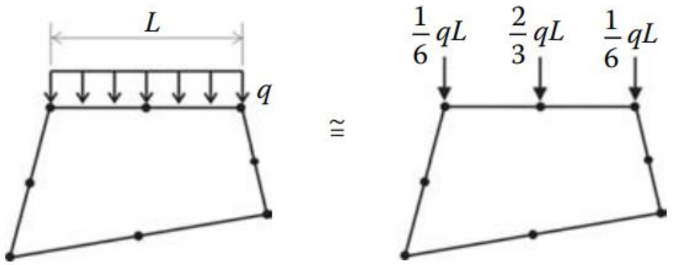
$$q_y = \left(q_n \frac{\partial x}{\partial \xi} + q_t \frac{\partial y}{\partial \xi} \right) d\xi$$

$$\int_{A_e} [N]^T \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} dA = t \int_{L_{3-4}} [N(\xi, +1)]^T \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} dl$$

$$= t \sum_{i=1}^{ngp} W_i [N(\xi_i, +1)]^T \begin{Bmatrix} \left(q_t \frac{\partial x(\xi_i, +1)}{\partial \xi} - q_n \frac{\partial y(\xi_i, +1)}{\partial \xi} \right) \\ \left(q_n \frac{\partial x(\xi_i, +1)}{\partial \xi} + q_t \frac{\partial y(\xi_i, +1)}{\partial \xi} \right) \end{Bmatrix}$$

f_e = eldof×1

8×2

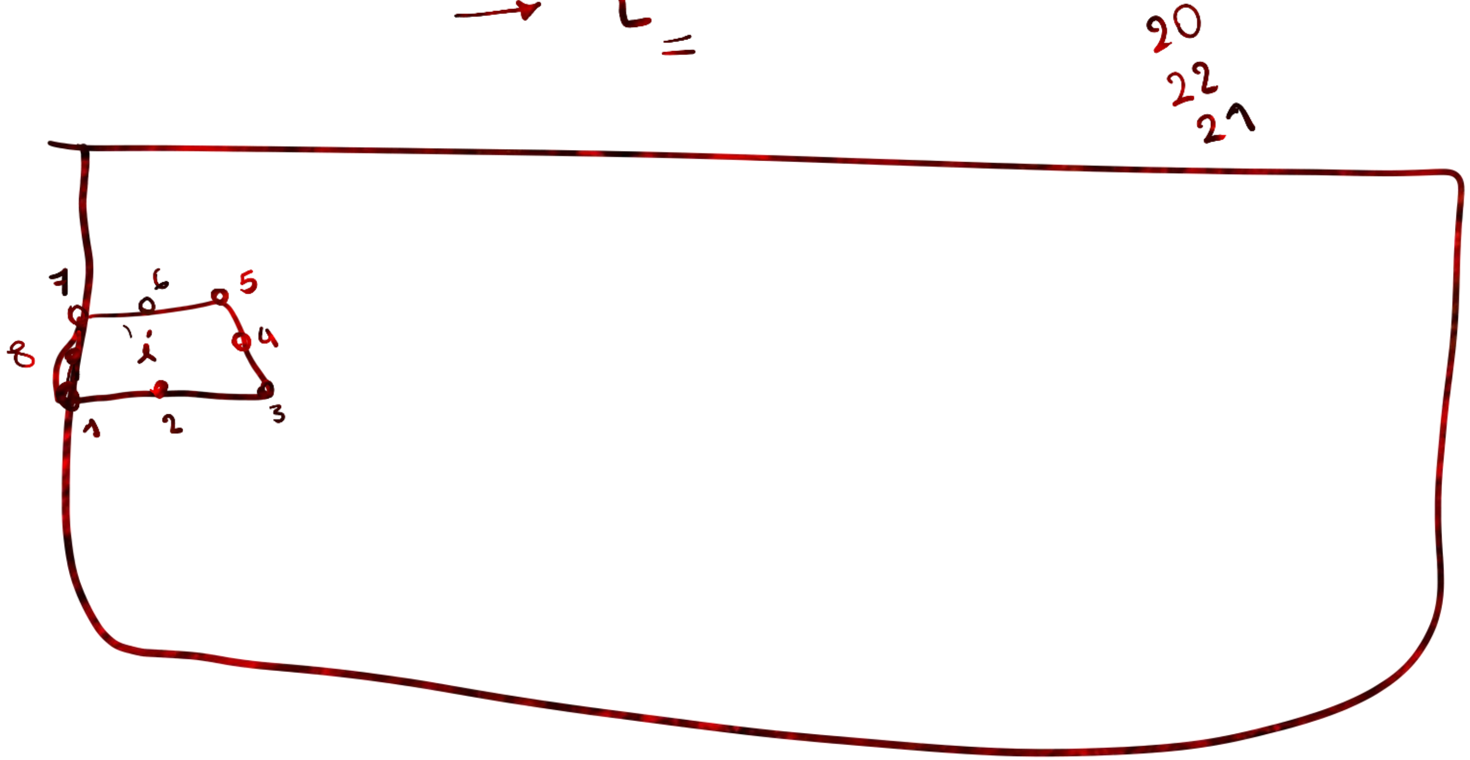


Concentrated Forces

$$\sum_{k=1} [N]_{x=x_k} \{P_k\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -P \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 2P \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 2P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

In practice, when the loads are uniformly distributed they are replaced by equivalent nodal loads. The preceding development is to be used only if the shape of the loading is complicated.

$i \rightarrow [20, 30, 25, 26, 27, 31, 22, 21]$



Plane Stress Problem: Q8

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

linsolve

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Plane Stress Problem: Q8

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

If $\{\delta_p\} = 0$ →

$$\{F_P\} = [K_{PF}]\{\delta_F\}$$

Plane Stress Problem: Q8

Calculation of the Element Resultants

Once the global system of equations is solved, we will compute the stresses at the centroid of the elements. For this we set ngp = 1.

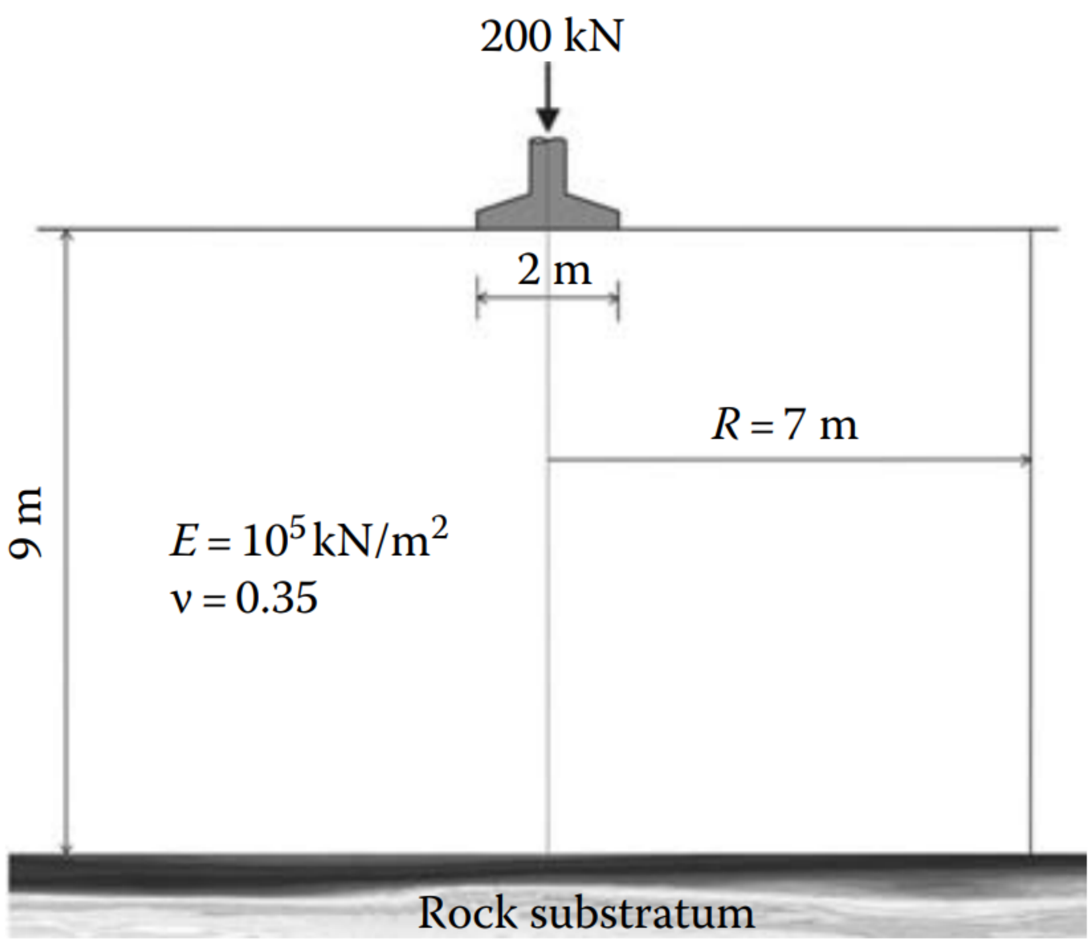
1. For each element
2. Retrieve the coordinates of its nodes coord(nne, 2) and its steering vector g(eldof) using the function elem_Q4.m
3. Retrieve its nodal displacements eld(eldof) from the global vector of displacements delta(n)

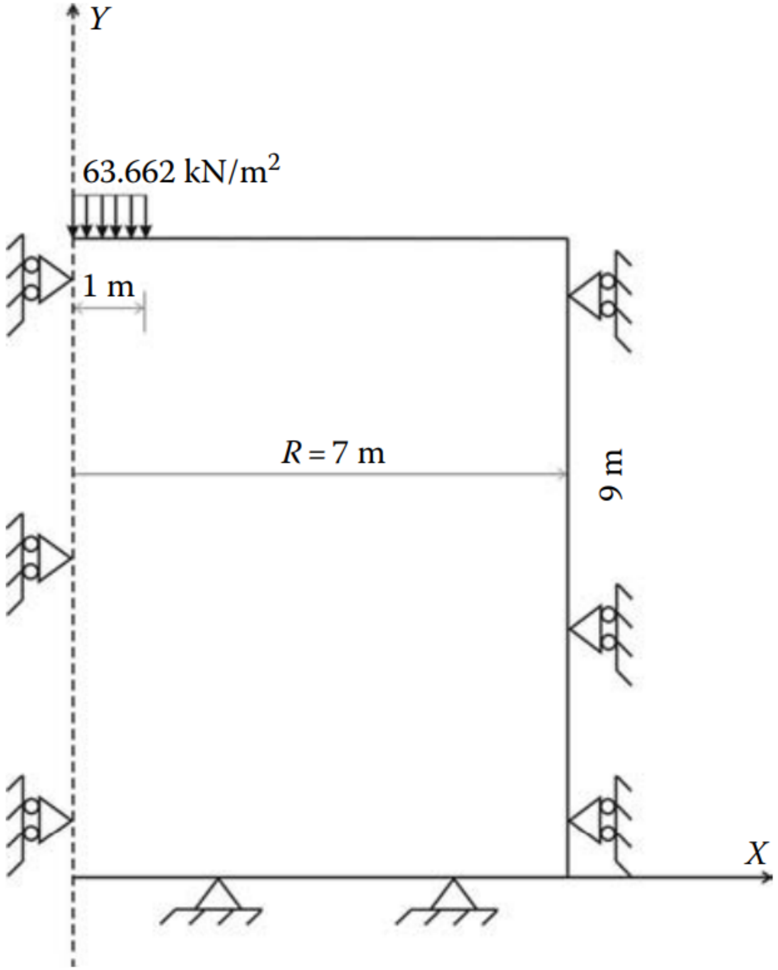
a. Loop over the Gauss points ig = 1 to ngp
b. Loop over the Gauss points jg = 1 to ngp
c. Use the function fmlin.m to compute the shape functions, vector fun, and their local derivatives, der, at the local coordinates $\xi = \text{samp}(ig, 1)$ and $\eta = \text{samp}(jg, 1)$
d. Evaluate the Jacobian jac = der ∗ coord
e. Evaluate the determinant of the Jacobian as d = det(jac)
f. Compute the inverse of the Jacobian as jac1 = inv(jac)
g. Compute the derivatives of the shape functions with respect to the global coordinates x and y as deriv = jac1 ∗ der
h. Use the function formbee.m to form the strain matrix bee
i. Compute the strains as eps = B ∗ eld
j. Compute the stresses as sigma = D ∗ eps

4. Store the stresses in the matrix SIGMA(nel, 3)

Axisymmetric Problem

Problem Discerption





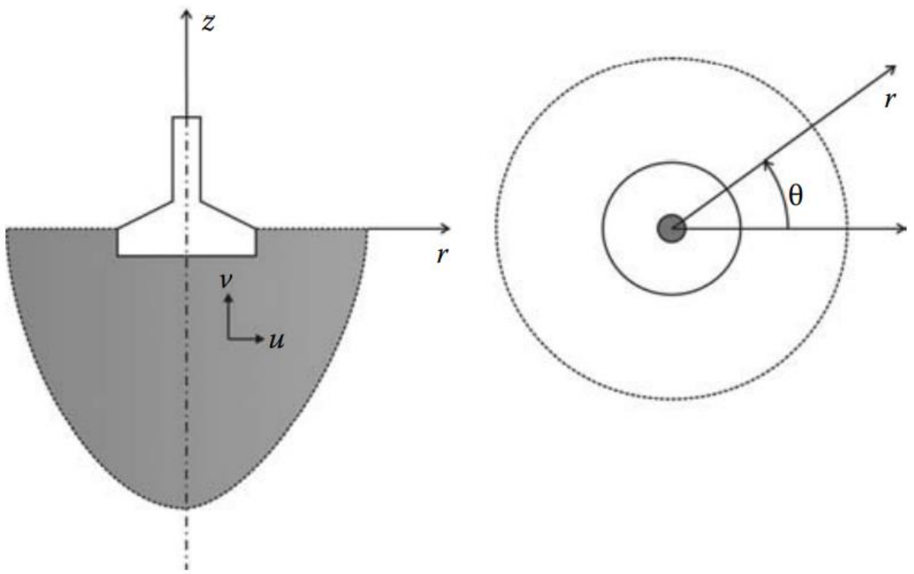
Axisymmetric Problem

LENGTH	MASS	TIME	FORCE	STRESS	ENERGY	VELOCITY	ACCELERATION
mm	ton	S	N	MPa	mJ	1e-03 m/s	1e-03 m/s^2
mm	kg	ms	kN	GPa	1e+03 mJ	m/s	1e+03 m/s^2
mm	g	ms	N	MPa	mJ	m/s	1e+03 m/s^2
mm	kg	S	mN	kPa	1e-03 mJ	1e-03 m/s	1e-03 m/s^2
mm	g	S	1e-06 N	Pa	1e-06 mJ	1e-03 m/s	1e-03 m/s^2
mm	kgf-s^2/mm	S	kgf	kgf/mm^2	kgf-mm	1e-03 m/s	1e-03 m/s^2
m	kg	S	N	Pa	J	m/s	m/s^2
cm	kg	S	1e-02 N	1e+02 Pa	1e-04. J	1e-02 m/s	1e-02 m/s^2
cm	kg	ms	1e+04 N	1e+08 Pa	1e+02 J	1e+01 m/s	1e+04 m/s^2
cm	kg	us	1e+10 N	1e+14 Pa	1e+08 J	1e+04 m/s	1e+10 m/s^2
cm	g	S	dyne	dyne/cm^2	erg	1e-02 m/s	1e-02 m/s^2
cm	g	ms	1e+01 N	bar	1e-01 J	1e+01 m/s	1e+04 m/s^2
cm	g	us	1e+07 N	Mbar	1e+05 J	1e+04 m/s	1e+10 m/s^2
in	lbf-s^2/in	S	lbf	psi	lbf-in	in/s	in/s^2
ft	slug	S	lbf	psf	lbf-ft	ft/s	ft/s^2

Axisymmetric Problem

An axisymmetric problem is a **three-dimensional** problem that can be solved using a **two-dimensional model** provided that it posses a **symmetry of revolution** in both **geometry**, **material properties** and **loading**, and it can lend itself to a cylindrical coordinate.

The only displacements required to define its behavior are the ones in the r and z directions, denoted by u and v, respectively. They are not a function of θ.



Axisymmetric Problem

Data Preparation (Create Input file)

Nodes Coordinates geom(nnd, dim=2)

Element Connectivity connec(nel, nne=8)

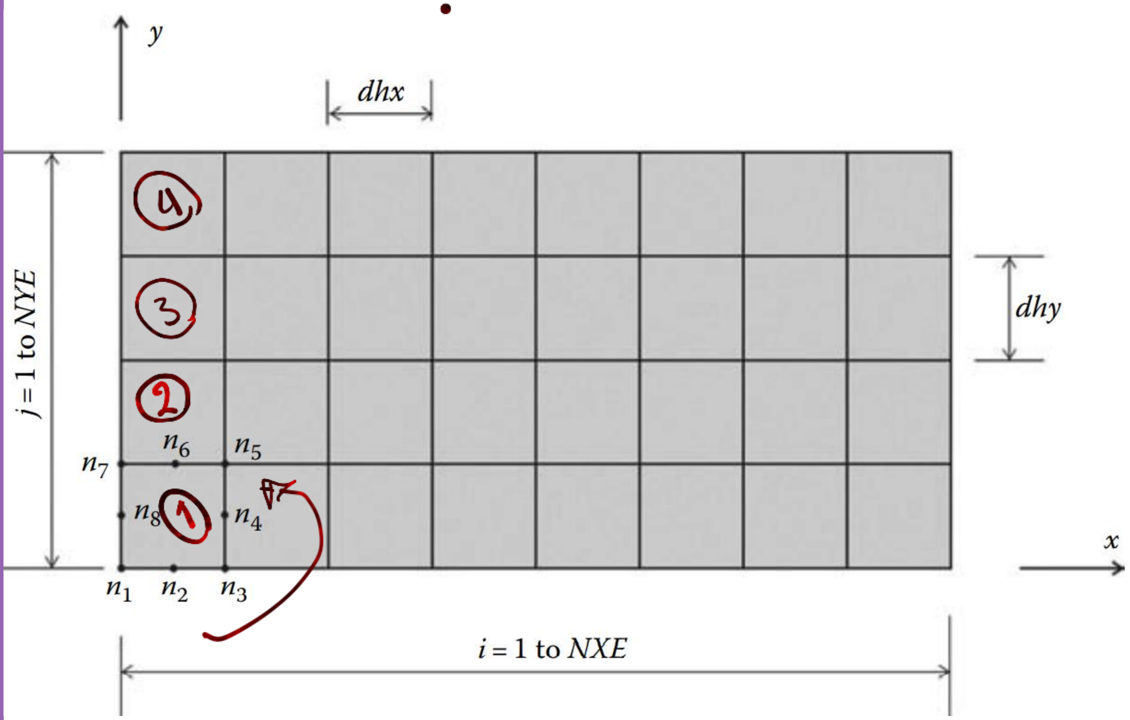
Material and Geometrical Properties $E = 10^5\ kPa \quad \nu = 0.35$

Boundary Conditions nf(nnd, nodof)

Loading The force in the global force vector **F**

Axisymmetric Problem

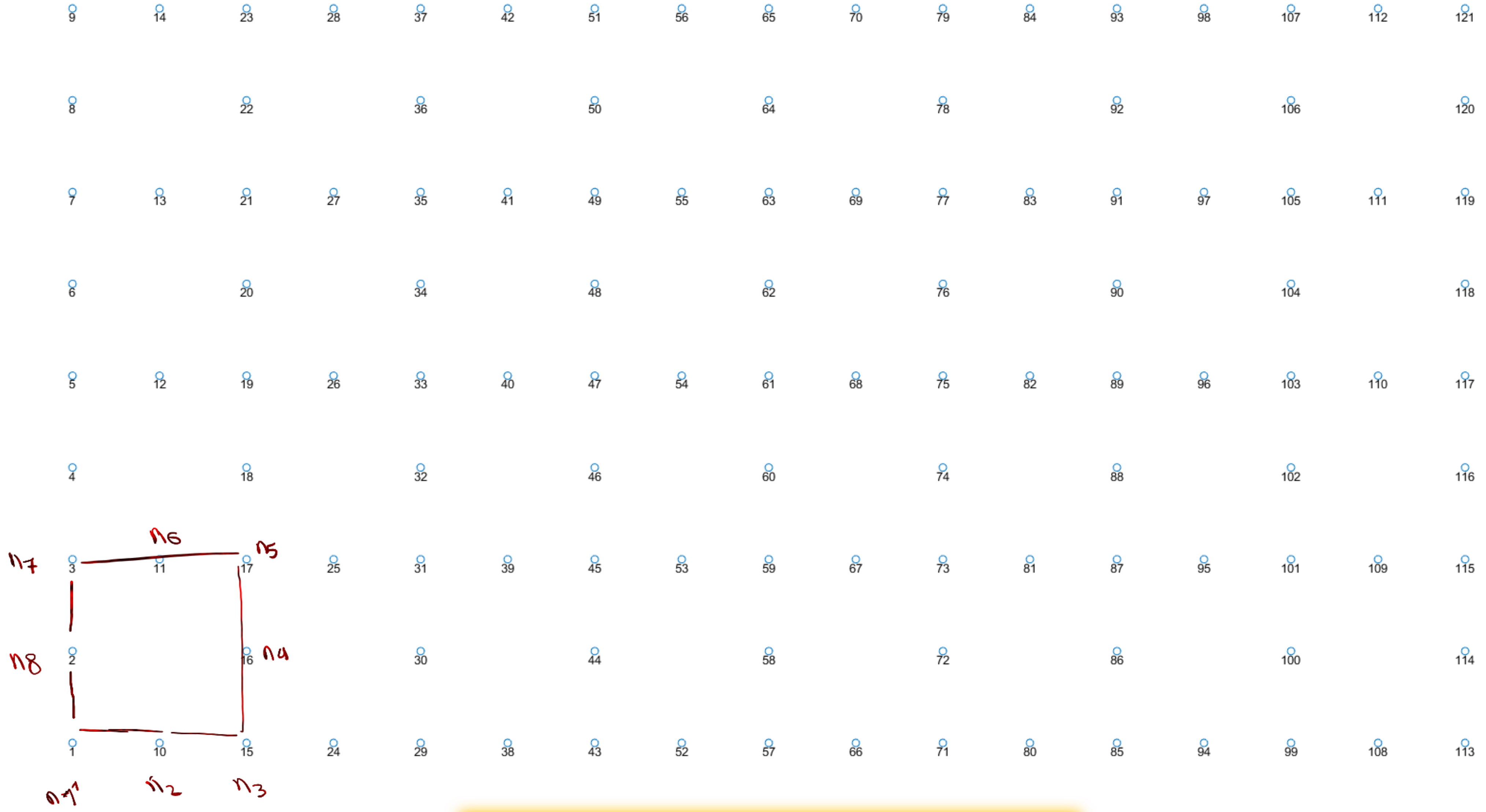
Discretization: Mesh Generation




```
nnd=0;
k=0;
for i=1:NXE
   for j=1:NYE
      k=k+1;
      n1=(i-1)*(3*NYE+2)+2*j - 1;
      n2=i*(3*NYE+2)+j - NYE - 1;
      n3=i*(3*NYE+2)+2*j-1;
      n4=n3 + 1;           n5=n3 + 2;            n6=n2 + 1;
      n7=n1 + 2;           n8=n1 + 1;
      geom(n1,:)=[(i-1)*dhx-X_origin,   (j-1)*dhy-Y_origin];
      geom(n3,:)=[i*dhx - X_origin,       (j-1)*dhy-Y_origin];
      geom(n5,:)=[i*dhx-X_origin,          j*dhy - Y_origin];
      geom(n7,:)=[(i-1)*dhx - X_origin,   j*dhy - Y_origin];
      geom(n2,:)=[(geom(n1,1)+geom(n3,1))/2 , (geom(n1,2)+geom(n3,2))/2];
     geom(n4,:)=[(geom(n3,1)+ geom(n5,1))/2 (geom(n3,2)+ geom(n5,2))/2];
      geom(n6,:)=[(geom(n5,1)+ geom(n7,1))/2 (geom(n5,2)+ geom(n7,2))/2];
      geom(n8,:)=[(geom(n1,1)+ geom(n7,1))/2 (geom(n1,2)+ geom(n7,2))/2];
      nel = k;
      nnd = n5;
      connec(k,:)=[n1 n2 n3 n4 n5 n6 n7 n8];
   end
end
```

Axisymmetric Problem

Discretization: Mesh Generation



Axisymmetric Problem

Interpolation

For an element having n nodes, the components of the displacement vector are interpolated using nodal approximations

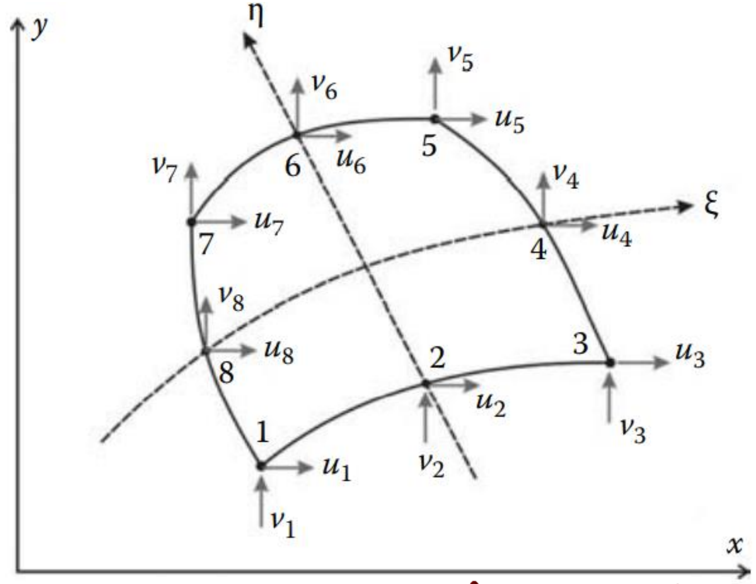
$$\begin{cases} u = N_1 u_1 + N_2 u_2 + \cdots + N_n u_n \\ v = N_1 v_1 + N_2 v_2 + \cdots + N_n v_n \end{cases}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \ldots & | & N_n & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \ldots & | & 0 & N_n \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

$$\{U\} = [N]\{a\}$$

Axisymmetric Problem

Interpolation



Eight-nodded Iso-parametric Element

$$\begin{Bmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ N_3(\xi,\eta) \\ N_4(\xi,\eta) \\ N_5(\xi,\eta) \\ N_6(\xi,\eta) \\ N_7(\xi,\eta) \\ N_8(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} -0.25(1-\xi)(1-\eta)(1+\xi+\eta) \\ 0.50(1-\xi^2)(1-\eta) \\ -0.25(1+\xi)(1-\eta)(1-\xi+\eta) \\ 0.50(1+\xi)(1-\eta^2) \\ -0.25(1+\xi)(1+\eta)(1-\xi-\eta) \\ 0.50(1-\xi^2)(1+\eta) \\ -0.25(1-\xi)(1+\eta)(1+\xi-\eta) \\ 0.50(1-\xi)(1-\eta^2) \end{Bmatrix}$$

$$u = c_0 + c_1\xi + c_2\eta + c_3\xi\eta + c_4\xi^2 + c_5\eta^2 + c_6\xi^2\eta + c_7\xi\eta^2$$

$$u(\xi=-1, \eta=-1) = u_1$$

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6 + N_7u_7 + N_8u_8$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 + N_5v_5 + N_6v_6 + N_7v_7 + N_8v_8$$

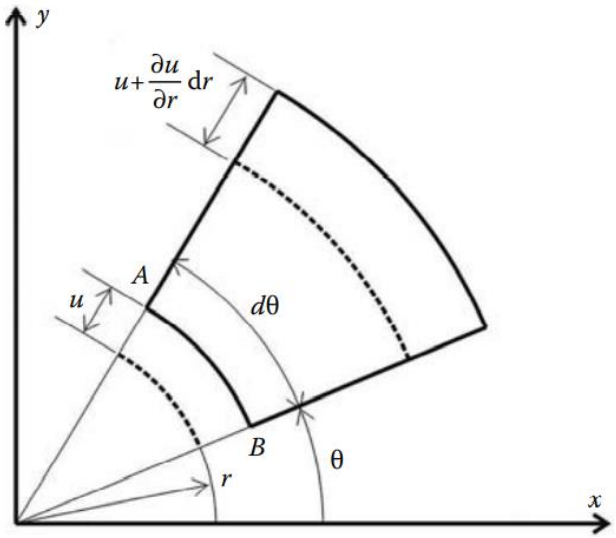
Axisymmetric Problem

Strain-Displacement Relations

The infinitesimal strain displacements relations for axisymmetric problems

$$\begin{cases} \epsilon_{rr} = \dfrac{\partial u}{\partial r} \\ \epsilon_{\theta} = \dfrac{(r+u)\,d\theta - r\,d\theta}{r\,d\theta} = \dfrac{u}{r} \\ \epsilon_{zz} = \dfrac{\partial v}{\partial z} \\ \gamma_{rz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial r} \end{cases}$$

$$\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta} \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ 1/r & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$



$$\{\epsilon\} = [L]U$$

$\sigma \rightarrow \varepsilon \rightarrow u \rightarrow u_i$

$\{U\} = [N]\{a\} \rightarrow \{\varepsilon\} = \underbrace{[L][N]}_{[B]}\{a\}$

Axisymmetric Problem

Strain-Displacement Relations

By substitution

$$\begin{cases} \{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}]\{\boldsymbol{U}\} \\ \{\boldsymbol{U}\} = [\boldsymbol{N}]\{\boldsymbol{a}\} \end{cases} \Rightarrow \{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}][\boldsymbol{N}]\{\boldsymbol{a}\} = [\boldsymbol{B}]\{\boldsymbol{a}\}$$

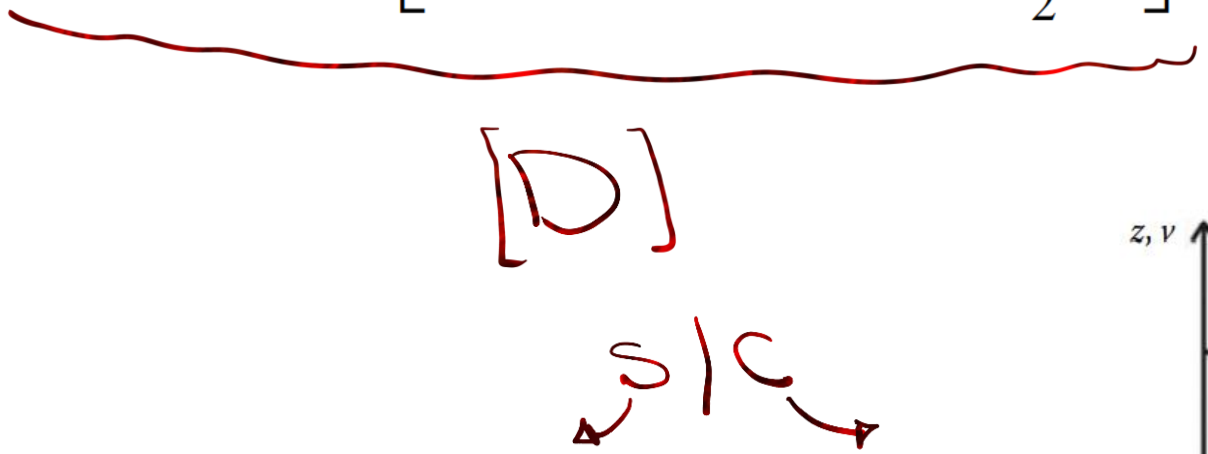
$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & | & \frac{\partial N_2}{\partial x} & 0 & | & \dots & | & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & | & 0 & \frac{\partial N_2}{\partial y} & | & \dots & | & 0 & \frac{\partial N_n}{\partial y} \\ \frac{N_1}{r} & 0 & | & \frac{N_2}{r} & 0 & | & \dots & | & \frac{N_n}{r} & 0 \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & | & \dots & | & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

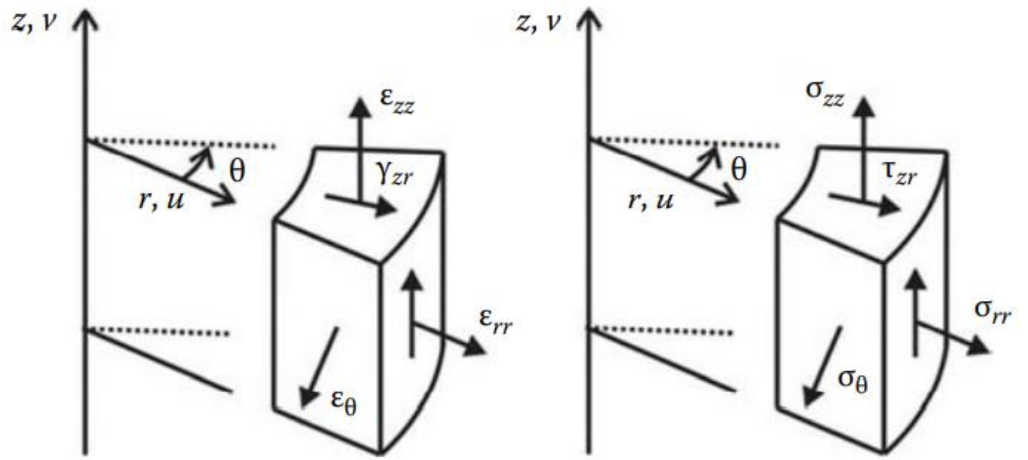
Axisymmetric Problem

Stress–Strain Relations

In an axisymmetric problem, the shear strains $\gamma_{r\theta}$ and $\gamma_{z\theta}$ and the shear stresses $\tau_{r\theta}$ and $\tau_{z\theta}$ all vanish because of the radial symmetry.

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta} \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta} \\ \gamma_{rz} \end{Bmatrix}$$





Axisymmetric Problem

Stiffness Matrix + Force Vectors

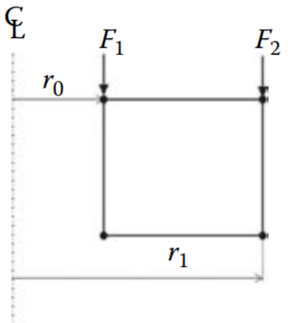
$$\left(\iiint_V \{B\}^T[D]\{B\}\,dV\right)\{a\} = \iiint_V \{B\}^T[D]\{\varepsilon_0\}dV - \iiint_V \{B\}^T\{\sigma_0\}dV + \iiint_V \{N\}^T\{F_b\}dV + \iint_S \{N\}^T\{T\}dS + \sum_{i=1}^{n}\{N\}^T\{F_p\}$$

$$[K_e] = \left[\int_{V_e}[B]^T[D][B]\,dv\right] = \left[\int\int\int_{V_e}[B]^T[D][B]r\,dr\,d\theta\,dz\right]$$

$$\{f_b\} = \int\int_{A_e}[N]^T\begin{Bmatrix} b_r \\ b_z \end{Bmatrix} r\,dr\,dz$$

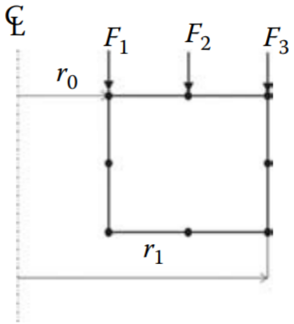
$$\{f_s\} = \int_L [N]^T\begin{Bmatrix} t_r \\ t_z \end{Bmatrix} r\,dl$$

$$\{f_c\} = \Sigma_i[N]^T r_i\begin{Bmatrix} P_r \\ P_z \end{Bmatrix}_i$$



$$F_1 = \frac{r_1 - r_0}{6}(2r_0 + r_1)$$

$$F_2 = \frac{r_1 - r_0}{6}(r_0 + 2r_1)$$



$$F_1 = \frac{r_1 - r_0}{6}r_0$$

$$F_2 = \frac{r_1 - r_0}{3}(r_0 + r_1)$$

$$F_3 = \frac{r_1 - r_0}{6}r_1$$

Axisymmetric Problem

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & \dots & | & N_8 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & \dots & | & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}$$

$$\{U\} = [N]\{a\}$$

$$\{\epsilon\} = [B]\{a\}$$

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & | & \dfrac{\partial N_2}{\partial x} & 0 & | & \dots & | & \dfrac{\partial N_n}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & | & 0 & \dfrac{\partial N_2}{\partial y} & | & \dots & | & 0 & \dfrac{\partial N_n}{\partial y} \\ \dfrac{N_1}{r} & 0 & | & \dfrac{N_2}{r} & 0 & | & \dots & | & \dfrac{N_n}{r} & 0 \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & | & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & | & \dots & | & \dfrac{\partial N_n}{\partial y} & \dfrac{\partial N_n}{\partial x} \end{bmatrix}$$

Axisymmetric Problem

Stiffness Matrix

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & | & N_2 & 0 & | & \dots & \dots & | & N_8 & 0 \\ 0 & N_1 & | & 0 & N_2 & | & \dots & \dots & | & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}$$

$$\{U\} = [N]\{a\} \quad \Rightarrow \quad \{\epsilon\} = [B]\{a\}$$

$$\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial N}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

deriv

coord

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$[J] = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \dots & \dfrac{\partial N_8}{\partial \xi} \\ \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \dots & \dfrac{\partial N_8}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_8 & y_8 \end{bmatrix}$$

$$x = \sum N_i x_i$$

$$\frac{\partial x}{\partial \xi} = \sum \frac{N_i}{\partial \xi} x_i$$

$$x = N_1 x_1 + N_2 x_2 + \cdots + N_8 x_8$$

$$y = N_1 y_1 + N_2 y_2 + \cdots + N_8 y_8$$

$$r = N_1 x_1 + N_2 x_2 + \cdots + N_8 x_8$$

Deriv

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

Axisymmetric Problem

Numerical Integration of the Stiffness Matrix

$$[K_e] = \int_{-1}^{+1}\int_{-1}^{+1} [B(\xi,\eta)]^T [D][B(\xi,\eta)] r(\xi,\eta) det[J(\xi,\eta)]\, d\eta\, d\xi$$

$$= \sum_{i=1}^{ngp}\sum_{j=1}^{ngp} W_i W_j [B(\xi_i,\eta_j)]^T [D][B(\xi_i,\eta_j)] r(\xi_i,\eta_j) det[J(\xi_i,\eta_j)]$$

For each element, it is evaluated as follows:

1. For every element i = 1 to nel
2. Retrieve the coordinates of its nodes coord(nne, 2) and its steering vector g(eldof) using the function elem_Q8.m
3. Initialize the stiffness matrix to zero a. Loop over the Gauss points ig = 1 to ngp b. Retrieve the weight wi as samp(ig, 2)

i. Loop over the Gauss points jg = 1 to ngp
ii. Retrieve the weight wj as samp(jg, 2)
iii. Use the function fmquad.m to compute the shape functions, vector fun, and their derivatives, matrix der, in local coordinates, ξ = samp(ig, 1) and η =samp(jg, 1).
iv. Evaluate the Jacobian jac = der ∗ coord
v. Evaluate the determinant of the Jacobian as d = det(jac)
vi. Compute the inverse of the Jacobian as jac1 = inv(jac)
vii. Compute the derivatives of the shape functions with respect to the global coordinates x and y as deriv = jac1 ∗ der
viii. Use the function formbee_axi to form the strain matrix bee and calculate the radius r at the integration point as $r = \sum_j^{nne} N_j x_j$
ix. Compute the stiffness matrix as $ke = ke + d * wi * wj * B^T * D * B * r$

4. Assemble the stiffness matrix ke into the global matrix kk

Axisymmetric Problem

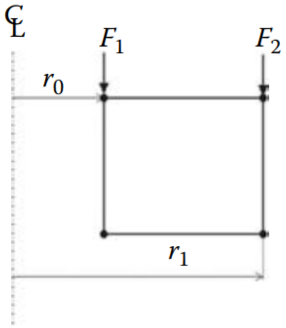
Force Vectors

Body Forces

$$\{f_b\} = \iint_{A_e} [N]^T \begin{Bmatrix} b_r \\ b_z \end{Bmatrix} r\, dr\, dz$$

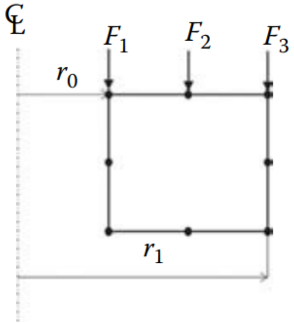
Traction Forces

$$\{f_s\} = \int_L [N]^T \begin{Bmatrix} t_r \\ t_z \end{Bmatrix} r\, dl$$



$$F_1 = \frac{r_1 - r_0}{6}(2r_0 + r_1)$$

$$F_2 = \frac{r_1 - r_0}{6}(r_0 + 2r_1)$$



$$F_1 = \frac{r_1 - r_0}{6} r_0$$

$$F_2 = \frac{r_1 - r_0}{3}(r_0 + r_1)$$

$$F_3 = \frac{r_1 - r_0}{6} r_1$$

Concentrated Forces

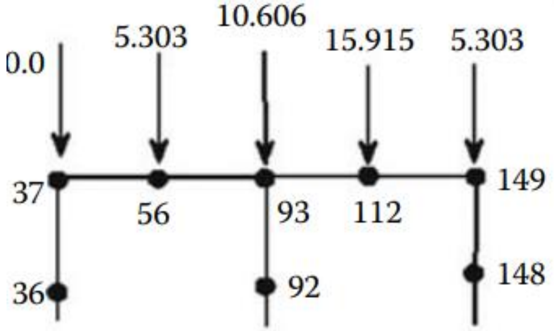
$$\{f_c\} = \Sigma_i [N]^T r_i \begin{Bmatrix} P_r \\ P_z \end{Bmatrix}_i$$

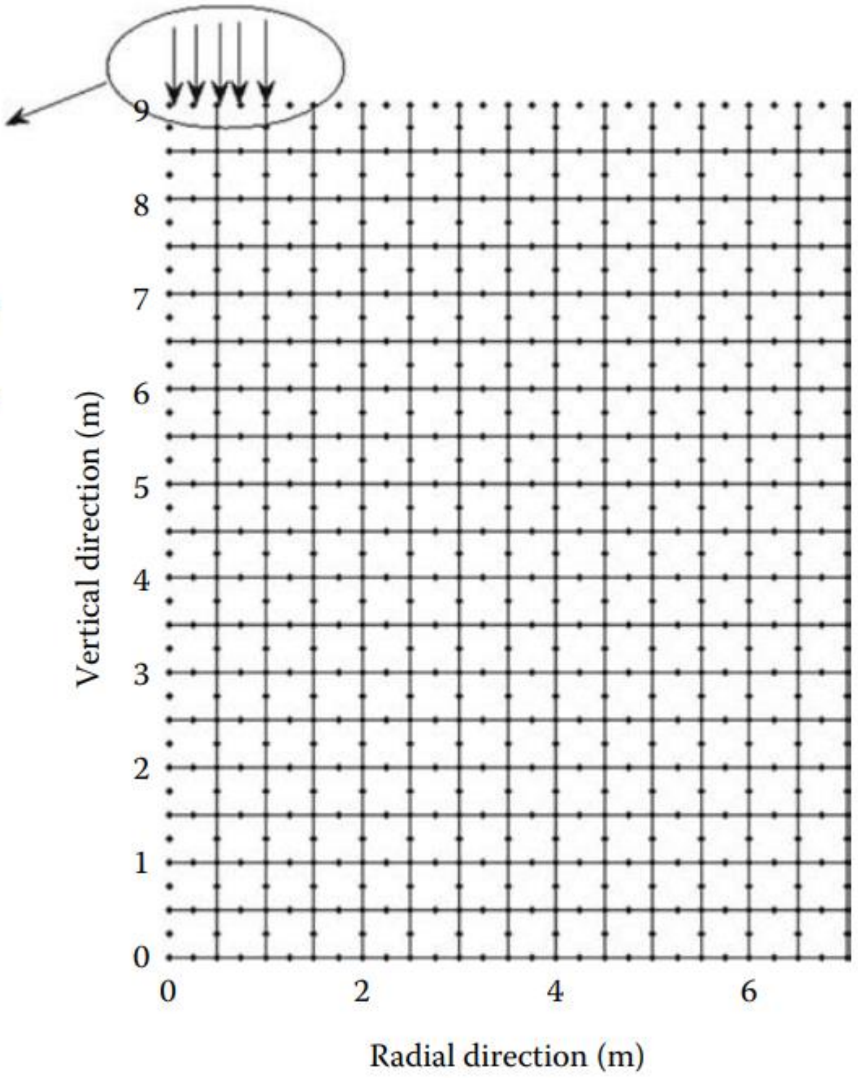
Axisymmetric Problem

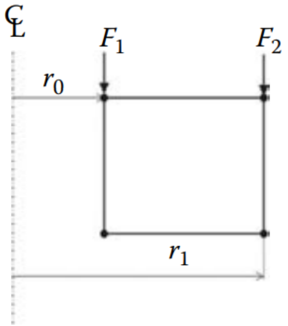
Discretization: Mesh Generation

$$\{f_s\} = \int_L [N]^T \begin{Bmatrix} t_r \\ t_z \end{Bmatrix} r\, dl$$

$$\{f_s\} = \int_L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ N_8 & 0 \\ 0 & N_8 \end{bmatrix} \begin{Bmatrix} 0 \\ 63662\ (N/m^2) \end{Bmatrix} r\, dl$$

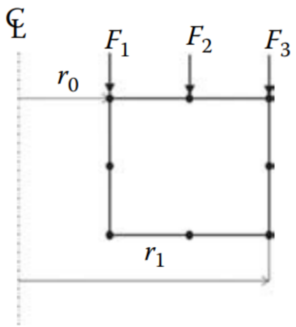






$$F_1 = \frac{r_1 - r_0}{6}(2r_0 + r_1)$$

$$F_2 = \frac{r_1 - r_0}{6}(r_0 + 2r_1)$$



$$F_1 = \frac{r_1 - r_0}{6} r_0$$

$$F_2 = \frac{r_1 - r_0}{3}(r_0 + r_1)$$

$$F_3 = \frac{r_1 - r_0}{6} r_1$$

Axisymmetric Problem

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Axisymmetric Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

If $\{\delta_p\} = 0$ →

$$\{F_P\} = [K_{PF}]\{\delta_F\}$$

Axisymmetric Problem

Calculation of the Element Resultants

Element Displacement

↓

B Matrix

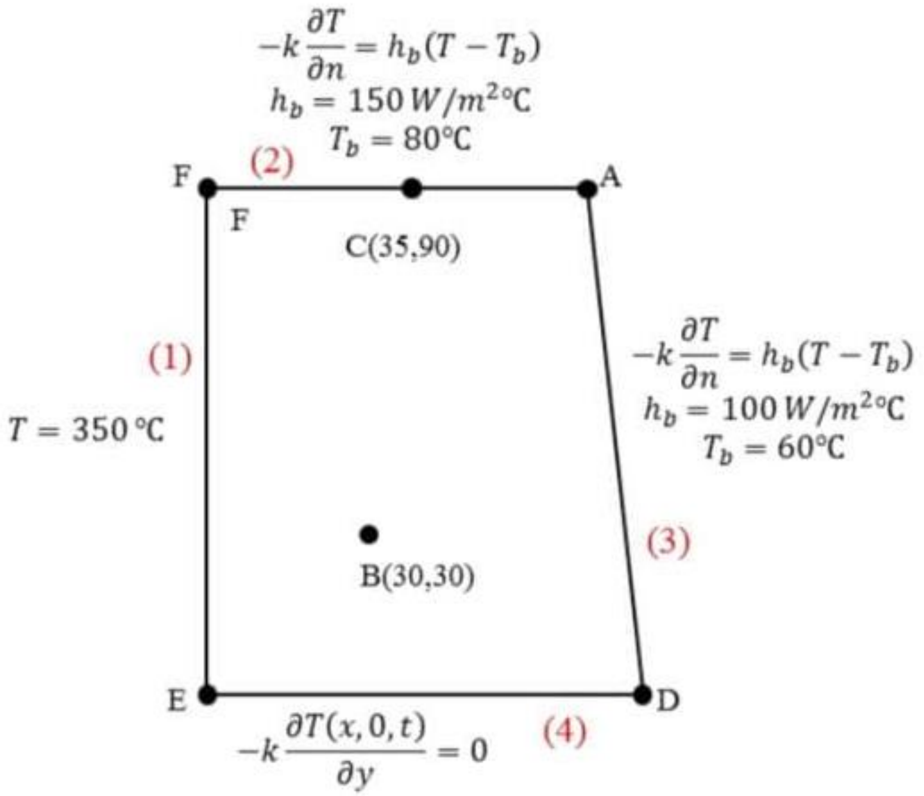
↓

Strain

↓

Stress

Problem: Transient Thermal Analysis



$Q = 100\frac{kW}{m^3}$

$c = 400 \quad \frac{J}{kg\text{°C}}$

$k = 40 \quad W/m\text{°C}$

$\rho = 7800\frac{kg}{m^3}$

$T(x,y,0) = 50\,\text{°C}$

$E:(0,0) \quad D:(80,0) \quad A:(70,90) \quad F:(0,90) \quad mm$

Problem: Transient Thermal Analysis

Finite Element Method = *Space* Discretization + Interpolation

Dynamic Problem (PDEs) → **FEM** → System of ODEs (Linear or Non-linear)

$$w(x,t) + \frac{\partial}{\partial x}\left(AE\frac{\partial u(x,t)}{\partial x}\right) = \rho\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$[M]\{\ddot{a}(t)\} + [K]\{a(t)\} = F(t) \xrightarrow{\textit{Time}\text{ Discretization}} \{a\}$$

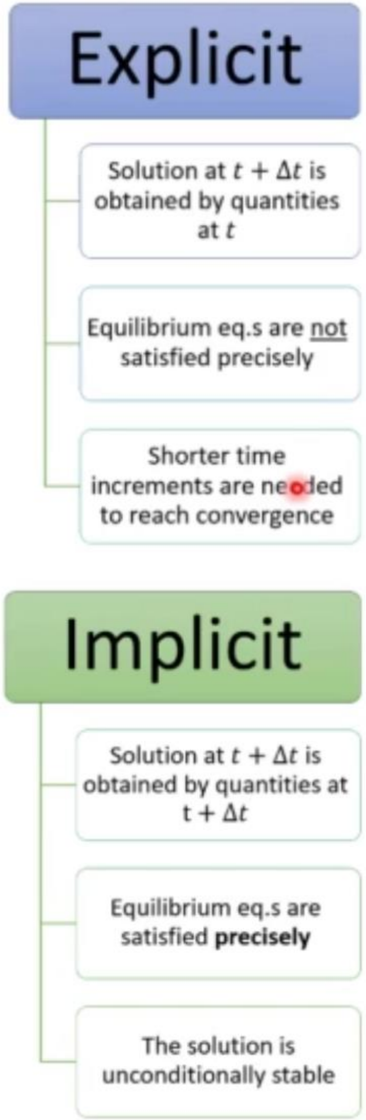
$$\{a(t_i)\} = \{a_i\}$$

$$\{a(t_i + \Delta t)\} = \{a_{i+1}\}$$

Problem: Transient Thermal Analysis

Time Discretization

- **Time Integration**
  - **Explicit**
    - Central Difference Method
    - Runge-Kutta Method
  - **Implicit**
    - Newmark-Beta Method
    - Wilson-Theta Method



Time Discretization

Explicit

Solution at $t + \Delta t$ is obtained by quantities at t
Equilibrium eq.s are not satisfied precisely
Shorter time increments are needed to reach convergence

- Central Difference Method
- Runge-Kutta Method

Implicit

Solution at $t + \Delta t$ is obtained by quantities at $t + \Delta t$
Equilibrium eq.s are satisfied precisely
The solution is unconditionally stable

- Newmark-Beta Method
- Wilson-Theta Method

Problem: Transient Thermal Analysis

Implicit Integration **vs.** Explicit Integration

Implicit Integration	Explicit Integration
$[M]\{x''\} + [C]\{x'\} + [K]\{x\} = \{f\}$	$[M]\{x''\} + [C]\{x'\} + [K]\{x\} = \{f\}$
$[K]\{x\} = \{f\} - ([M]\{x''\} + [C]\{x'\})$	$[M]\{x''\} = \{f\} - ([C]\{x'\} + [K]\{x\})$
$[K]^{-1}[K]\{x\} = [K]^{-1}(\{f\} - ([M]\{x''\} + [C]\{x'\}))$	$[M]^{-1}[M]\{x''\} = [M]^{-1}(\{f\} - ([C]\{x'\} + [K]\{x\}))$
$\{x\} = [K]^{-1}(\{f\} - ([M]\{x''\} + [C]\{x'\}))$	$\{x''\} = [M]^{-1}(\{f\} - ([C]\{x'\} + [K]\{x\}))$

Problem: Transient Thermal Analysis

Explicit Method: **Central Difference Method**

$$[M]\{\ddot{d}_i\} + [K]\{d_i\} = \{F_i\}$$

$$\{\dot{d}_i\} = \frac{\{d_{i+1}\} - \{d_{i-1}\}}{2(\Delta t)}$$

$$\{\ddot{d}_i\} = \frac{\{\dot{d}_{i+1}\} - \{\dot{d}_{i-1}\}}{2(\Delta t)}$$

$$\Rightarrow \{\ddot{d}_i\} = \frac{\{d_{i+1}\} - 2\{d_i\} + \{d_{i-1}\}}{(\Delta t)^2}$$

$$[M]\{d_{i+1}\} = (\Delta t)^2\{F_i\} + [2[M] - (\Delta t)^2[K]]\{d_i\} - [M]\{d_{i-1}\}$$

$$\{d_{i-1}\} = \{d_i\} - (\Delta t)\{\dot{d}_i\} + \frac{(\Delta t)^2}{2}\{\ddot{d}_i\}$$

d_0

d_{-1} → taylor expansion, $i=0$

d_{-1}

Problem: Transient Thermal Analysis

Step 1

Given: $\{d_0\}$, $\{\dot{d}_0\}$, and $\{F(t)\}$.

Step 2

If $\{\ddot{d}_0\}$ is not initially given, solve $\{\ddot{d}_0\} = [M]^{-1}(\{F_0\} - [K]\{d_0\})$ at $t = 0$ for $\{\ddot{d}_0\}$

Step 3

By using Taylor expansion, obtain is $\{d_{-1}\}$; that is,

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \frac{(\Delta t)^2}{2}\{\ddot{d}_0\}$$

Step 4

now solve equation for $\{d_1\}$

$$\{d_1\} = [M]^{-1}\{(\Delta t)^2\{F_0\} + [2[M] - (\Delta t)^2[K]]\{d_0\} - [M]\{d_{-1}\}\}$$

Step 5

solve for $\{\ddot{d}_1\}$ as

$$\{\ddot{d}_1\} = [M]^{-1}(\{F_1\} - [K]\{d_1\})$$

Problem: Transient Thermal Analysis

Step 6

With $\{d_0\}$ initially given, and $\{d_1\}$ determined from step 4, use Eq. below to obtain $\{d_2\}$

$$\{d_2\} = [M]^{-1}\{(\Delta t)^2\{F_1\} + [2[M] - (\Delta t)^2[K]]\{d_1\} - [M]\{d_0\}\}$$

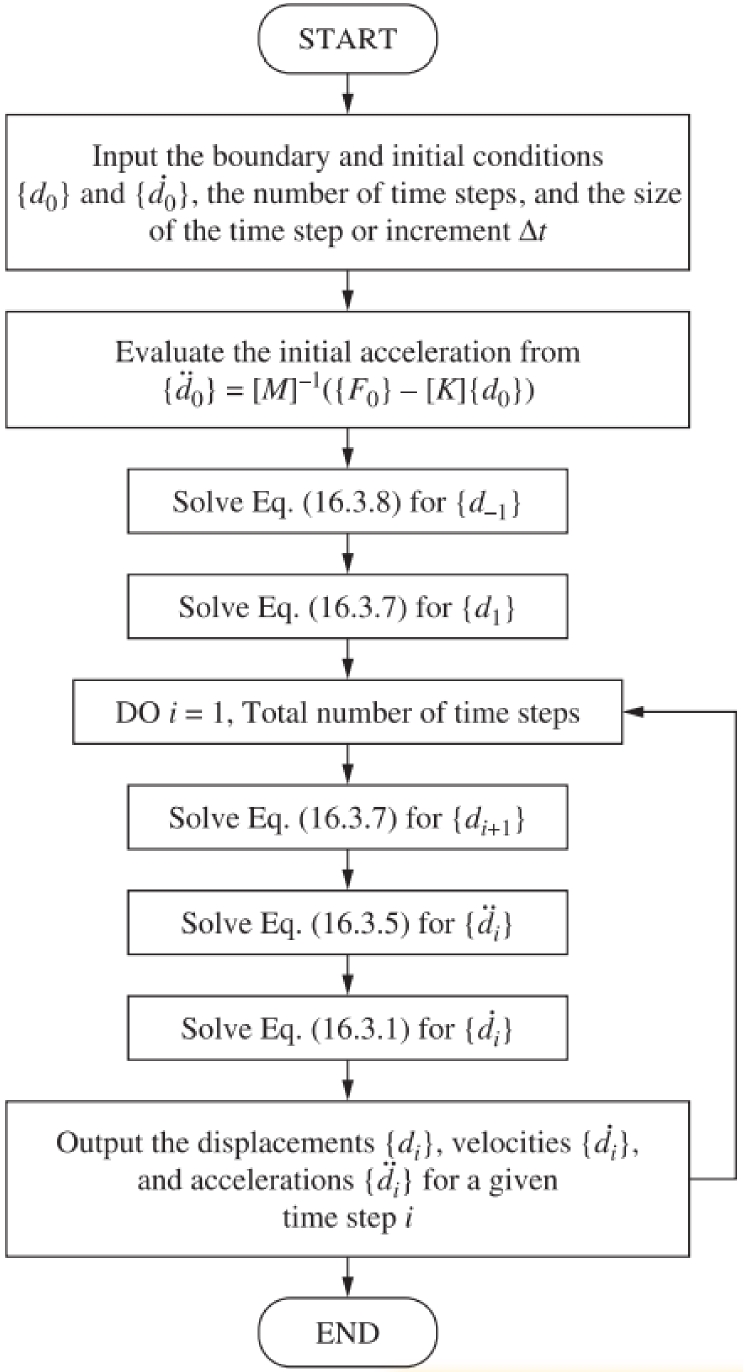
Step 7

Using the result of step 5 and initial condition $\{d_0\}$ given in step 1, determine the velocity at the first time step by Eq below

$$\{\dot{d}_1\} = \frac{\{d_2\} - \{d_0\}}{2(\Delta t)}$$

Step 8

Use steps 5 through 7 repeatedly to obtain the displacement, acceleration, and velocity for all other time steps.



$$\{d_{i-1}\} = \{d_i\} - (\Delta t)\{\dot{d}_i\} + \frac{(\Delta t)^2}{2}\{\ddot{d}_i\} \tag{16.3.8}$$

$$[M]\{d_{i+1}\} = (\Delta t)^2\{F_i\} + [2[M] - (\Delta t)^2[K]]\{d_i\} - [M]\{d_{i-1}\} \tag{16.3.7}$$

$$[M]\{d_{i+1}\} = (\Delta t)^2\{F_i\} + [2[M] - (\Delta t)^2[K]]\{d_i\} - [M]\{d_{i-1}\} \tag{16.3.7}$$

$$\{\ddot{d}_i\} = [M]^{-1}(\{F_i\} - [K]\{d_i\}) \tag{16.3.5}$$

$$\{\dot{d}_i\} = \frac{\{d_{i+1}\} - \{d_{i-1}\}}{2(\Delta t)} \tag{16.3.1}$$

Problem: Transient Thermal Analysis

Implicit Method: **Newmark's Method**

$$[M]\{\ddot{d}_i\} + [K]\{d_i\} = \{F_i\} \quad \rightarrow \quad [M]\{\ddot{d}_{i+1}\} = \{F_{i+1}\} - [K]\{d_{i+1}\}$$

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t)[(1-\gamma)\{\ddot{d}_i\} + \gamma\{\ddot{d}_{i+1}\}]$$

The parameter β is generally chosen between 0 and $\frac{1}{4}$, and γ is often taken to be $\frac{1}{2}$.

$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2[(\tfrac{1}{2} - \beta)\{\ddot{d}_i\} + \beta\{\ddot{d}_{i+1}\}] \quad \times \quad [M] \quad \rightarrow$$

$$[M]\{d_{i+1}\} = [M]\{d_i\} + (\Delta t)[M]\{\dot{d}_i\} + (\Delta t)^2[M](\tfrac{1}{2} - \beta)\{\ddot{d}_i\} + \beta(\Delta t)^2[\{F_{i+1}\} - [K]\{d_{i+1}\}]$$

$$([M] + \beta(\Delta t)^2[K])\{d_{i+1}\} = \beta(\Delta t)^2\{F_{i+1}\} + [M]\{d_i\} + (\Delta t)[M]\{\dot{d}_i\} + (\Delta t)^2[M](\tfrac{1}{2} - \beta)\{\ddot{d}_i\}$$

$$\underbrace{\left(\frac{[M]}{\beta(\Delta t)^2} + [K]\right)}_{[K']}\{d_{i+1}\} = \underbrace{\{F_{i+1}\} + \frac{[M]}{\beta(\Delta t)^2}\{d_i\} + \frac{[M]}{\beta(\Delta t)}\{\dot{d}_i\} + \frac{[M]}{\beta}\left(\frac{1}{2} - \beta\right)\{\ddot{d}_i\}}_{\{F'_{i+1}\}}$$

$$[K']\{d_{i+1}\} = \{F'_{i+1}\} \quad \rightarrow$$

Problem: Transient Thermal Analysis

Step 1

Starting at time $t = 0$, $\{d_0\}$ and $\{\dot{d}_0\}$ is known from the given initial conditions.

Step 2

Solve Eq. below at $t = 0$ for $\{\ddot{d}_0\}$; that is,

$$\{\ddot{d}_0\} = [M]^{-1}(\{F_0\} - [K]\{d_0\})$$

Step 3

Solve Eq. below for $\{d_1\}$, because $\{F'_{i+1}\}$ is known for all time steps and $\{d_0\}$, $\{\dot{d}_0\}$, and $\{\ddot{d}_0\}$ are now known from steps 1 and 2.

$$[K']\{d_{i+1}\} = \{F'_{i+1}\}$$

Step 4

Use Eq. below to solve for $\{\ddot{d}_1\}$ as

$$\{\ddot{d}_1\} = \frac{1}{\beta(\Delta t)^2}\left[\{d_1\} - \{d_0\} - (\Delta t)\{\dot{d}_0\} - (\Delta t)^2\left(\frac{1}{2} - \beta\right)\{\ddot{d}_0\}\right]$$

Step 5

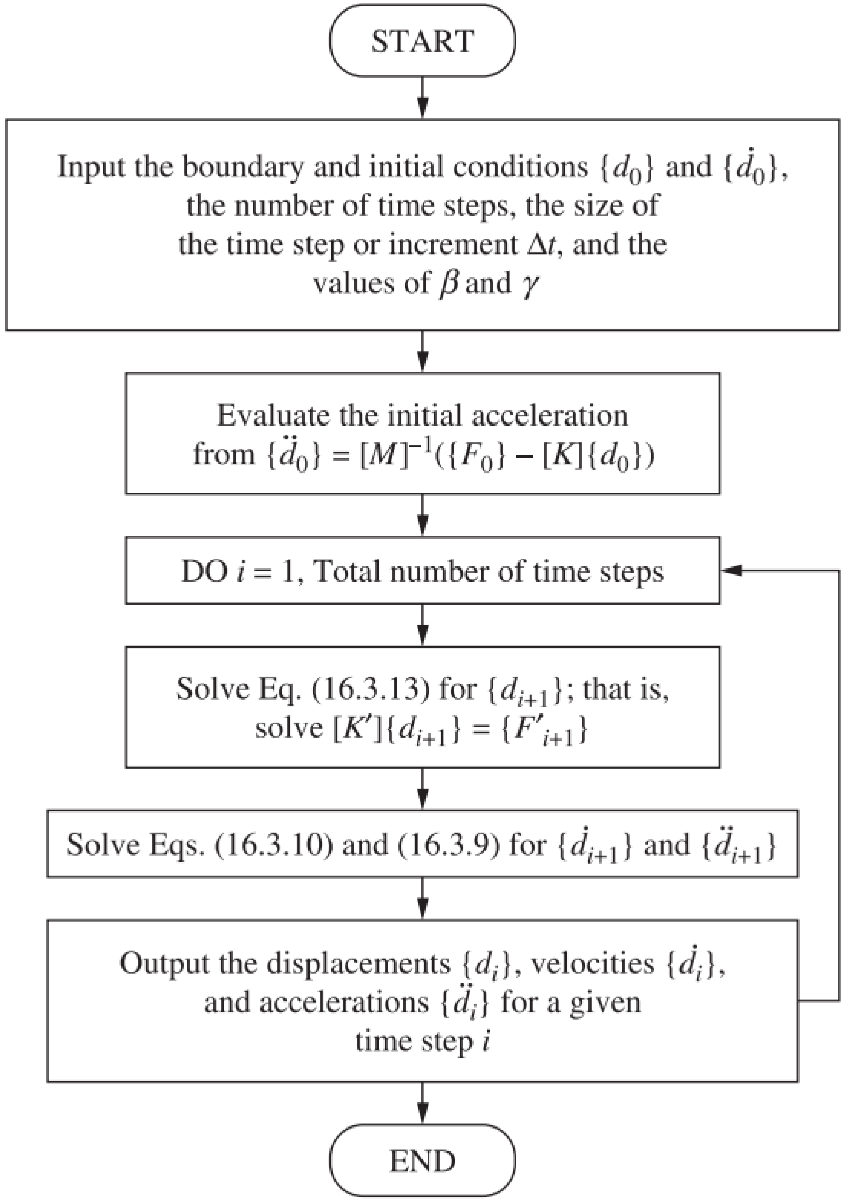
Solve Eq. below directly for $\{\dot{d}_1\}$

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t)[(1 - \gamma)\{\ddot{d}_i\} + \gamma\{\ddot{d}_{i+1}\}]$$

Step 6

Using the results of steps 4 and 5, go back to step 3 to solve for $\{d_2\}$ and then to steps 4 and 5 to solve for $\{\ddot{d}_2\}$ and $\{\dot{d}_2\}$. Use steps 3–5 repeatedly to solve for $\{d_{i+1}\}$, $\{\dot{d}_{i+1}\}$, and $\{\ddot{d}_{i+1}\}$

Problem: Transient Thermal Analysis



$$[K']\{d_{i+1}\} = \{F'_{i+1}\}$$

$[K']$ $\qquad$ $\{F'_{i+1}\}$

$$\underbrace{\left(\frac{[M]}{\beta(\Delta t)^2} + [K]\right)}_{[K']}\{d_{i+1}\} = \underbrace{\{F_{i+1}\} + \frac{[M]}{\beta(\Delta t)^2}\{d_i\} + \frac{[M]}{\beta(\Delta t)}\{\dot{d}_i\} + \frac{[M]}{\beta}\left(\frac{1}{2} - \beta\right)\{\ddot{d}_i\}}_{\{F'_{i+1}\}}$$

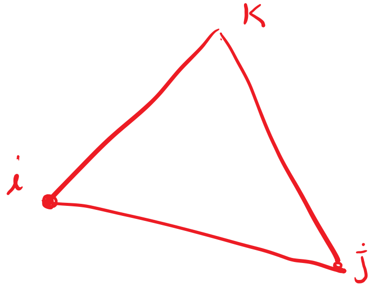
$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t)[(1-\gamma)\{\ddot{d}_i\} + \gamma\{\ddot{d}_{i+1}\}] \qquad (16.3.9)$$

$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2[(\tfrac{1}{2} - \beta)\{\ddot{d}_i\} + \beta\{\ddot{d}_{i+1}\}] \qquad (16.3.10)$$

Problem: Transient Thermal Analysis

Governing Differential Equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+Q=\rho c\frac{\partial T}{\partial t}$$



$$T(x,y,t)=c_1(t)+c_2(t)x+c_3(t)y$$

$$T=\begin{bmatrix}1 & x & y\end{bmatrix}\begin{bmatrix}c_1(t)\\c_2(t)\\c_3(t)\end{bmatrix}$$

$$T_i=c_1+c_2x_i+c_3y_i$$

$$T_j=c_1+c_2x_j+c_3y_j$$

$$T_k=c_1+c_2x_k+c_3y_k$$

$$\begin{bmatrix}T_i(t)\\T_j(t)\\T_k(t)\end{bmatrix}=\begin{bmatrix}1 & x_i & y_i\\1 & x_j & y_j\\1 & x_k & y_k\end{bmatrix}\begin{bmatrix}c_1(t)\\c_2(t)\\c_3(t)\end{bmatrix}$$

$$T(x,y,t)=\begin{bmatrix}1 & x & y\end{bmatrix}\begin{bmatrix}1 & x_i & y_i\\1 & x_j & y_j\\1 & x_k & y_k\end{bmatrix}^{-1}\begin{bmatrix}T_i(t)\\T_j(t)\\T_k(t)\end{bmatrix}$$

$$N_i(x,y)=m_{11}+m_{21}x+m_{31}y$$

$$N_j(x,y)=m_{12}+m_{22}x+m_{32}y$$

$$N_k(x,y)=m_{13}+m_{23}x+m_{33}y$$

$$T(x,y,t)=N_i(x,y)T_i(t)+N_j(x,y)T_j(t)+N_k(x,y)T_k(t)$$

$$\{T(t)\}=[\,T_1(t)\quad T_2(t)\quad T_3(t)\,]^T$$

$$[N]=[N_1\quad N_2\quad N_3]$$

Problem: Transient Thermal Analysis

$$T(x,y,t) = N_i(x,y)T_i(t) + N_j(x,y)T_j(t) + N_k(x,y)T_k(t)$$

$$\{T(t)\} = [\, T_1(t) \quad T_2(t) \quad T_3(t)\,]^T \qquad [N] = [N_1 \quad N_2 \quad N_3]$$

$$N_i(x,y) = m_{11} + m_{21}x + m_{31}y$$

$$N_j(x,y) = m_{12} + m_{22}x + m_{32}y$$

$$N_k(x,y) = m_{13} + m_{23}x + m_{33}y$$

$$m_{11} = (x_j y_k - x_k y_i)/2A \qquad m_{21} = (y_j - y_k)/2A \qquad m_{31} = (x_k - x_j)/2A$$

$$m_{12} = (x_k y_i - x_i y_k)/2A \qquad m_{22} = (y_k - y_i)/2A \qquad m_{32} = (x_i - x_k)/2A$$

$$m_{13} = (x_i y_j - x_j y_i)/2A \qquad m_{23} = (y_i - y_j)/2A \qquad m_{31} = (x_j - x_i)/2A$$

$$A = \frac{1}{2} det\left(\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}\right)$$

Problem: Transient Thermal Analysis

Weighted Residual Approach

$$\int u\,dv = uv\big| - \int v\,du$$

$$\iint_{A^e} \mathbf{N}^T \left[\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + Q - \rho c\,\frac{\partial T}{\partial t} \right] dxdy = 0$$

$$\int_{C^e} \mathbf{N}^T k \frac{\partial T}{\partial x} n_x dC - \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial x} k\, \frac{\partial T}{\partial x} dxdy + \int_{C^e} \mathbf{N}^T k \frac{\partial T}{\partial y} n_y dC - \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial y} k\, \frac{\partial T}{\partial y} dxdy + \iint_{A^e} \mathbf{N}^T Q dxdy - \iint_{A^e} \mathbf{N}^T \rho c \frac{\partial T}{\partial t} dxdy = 0$$

$$\iint_{A^e} \mathbf{N}^T \rho c \frac{\partial T}{\partial t} dxdy + \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial x} k\, \frac{\partial T}{\partial x} dxdy + \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial y} k\, \frac{\partial T}{\partial y} dxdy = \iint_{A^e} \mathbf{N}^T Q dxdy - \int_{C^e} \mathbf{N}^T q_n dC$$

$$\Rightarrow \quad \int_{C^e} \mathbf{N}^T q_n dC = \int_{FA} \mathbf{N}^T h_{FA} (T - T_{a\,FA}) dC + \int_{AD} \mathbf{N}^T h_{AD} (T - T_{a\,AD}) dC$$

$$\iint_{A^e} \mathbf{N}^T \rho c \frac{\partial T}{\partial t} dxdy + \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial x} k\, \frac{\partial T}{\partial x} dxdy + \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial y} k\, \frac{\partial T}{\partial y} dxdy + \int_{FA} \mathbf{N}^T h_{FA} T \Big|_{FA} dC + \int_{AD} \mathbf{N}^T h_{AD} T \Big|_{AD} dC$$
$$= \iint_{A^e} \mathbf{N}^T Q dxdy + \int_{FA} \mathbf{N}^T h_{FA} T_{a\,FA} dC + \int_{AD} \mathbf{N}^T h_{AD} T_{a\,AD} dC$$

Problem: Transient Thermal Analysis

$$\iint_{A^e} \mathbf{N}^T \rho c \frac{\partial T}{\partial t} dxdy + \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial x} k \frac{\partial T}{\partial x} dxdy + \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial y} k \frac{\partial T}{\partial y} dxdy + \int_{FA} \mathbf{N}^T h_{FA} T \Big|_{FA} dC + \int_{AD} \mathbf{N}^T h_{AD} T \Big|_{AD} dC$$
$$= \iint_{A^e} \mathbf{N}^T Q dxdy + \int_{FA} \mathbf{N}^T h_{FA} T_{a\,FA} dC + \int_{AD} \mathbf{N}^T h_{AD} T_{a\,AD} dC$$

$+$ $T = \mathbf{Na^e}$ $=$

$$\mathbf{C}^e \dot{\boldsymbol{a}}^e + \mathbf{K}^e \boldsymbol{a}^e = \mathbf{f}^e$$

$$\mathbf{K}^e = \mathbf{K}^e_{xx} + \mathbf{K}^e_{yy} + \mathbf{K}^e_{cvB}$$

$$\mathbf{K}^e_{xx} = \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial x} k \frac{\partial \mathbf{N}}{\partial x} dxdy$$

$+$

$$\mathbf{K}^e_{yy} = \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial y} k \frac{\partial \mathbf{N}}{\partial y} dxdy$$

$+$

$$\mathbf{K}^e_{cvB} = \int_F^A \mathbf{N}^T h_{FA} \mathbf{N}\, dC + \int_D^A \mathbf{N}^T h_{AD}\, \mathbf{N}\, dC$$

$$\mathbf{C}^e$$

$$\mathbf{C}^e = \iint_{A^e} \mathbf{N}^T \rho c\, \mathbf{N}\, dxdy$$

$$\mathbf{f}^e = \mathbf{f}^e_{\mathbf{Q}} + \mathbf{f}^e_{\mathbf{q}} + \mathbf{f}^e_{cvB}$$

$$\mathbf{f}^e_{\mathbf{Q}} = \iint_{A^e} \mathbf{N}^T Q dxdy$$

$+$

$$\mathbf{f}^e_{cvB} = \int_F^A \mathbf{N}^T h_{FA} T_{a\,FA} dC + \int_A^D \mathbf{N}^T h_{AD} T_{a\,AD}\, dC$$

Problem: Transient Thermal Analysis

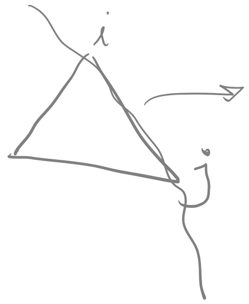
$$\mathbf{C}^e = \iint_{A^e} \mathbf{N}^T \rho c\, \mathbf{N}\, dxdy = \iint_{A^e} \begin{bmatrix} L_i \\ L_j \\ L_k \end{bmatrix} \rho c \begin{bmatrix} L_i & L_j & L_k \end{bmatrix} dxdy = \rho c \iint_{A^e} \begin{bmatrix} L_i^2 & L_i L_j & L_i L_k \\ L_j L_i & L_j^2 & L_j L_k \\ L_k L_i & L_k L_j & L_k^2 \end{bmatrix} dxdy = \frac{\rho c}{12} A_e \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\mathbf{K}_{xx}^e = \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial x} k \frac{\partial \mathbf{N}}{\partial x} dxdy = \iint_{A^e} \begin{bmatrix} m_{21} \\ m_{22} \\ m_{23} \end{bmatrix} k \begin{bmatrix} m_{21} & m_{22} & m_{23} \end{bmatrix} dxdy = kA_e \begin{bmatrix} m_{21}^2 & m_{21} m_{22} & m_{21} m_{23} \\ m_{22} m_{21} & m_{22}^2 & m_{22} m_{23} \\ m_{23} m_{21} & m_{23} m_{22} & m_{23}^2 \end{bmatrix}$$

$$\mathbf{K}_{yy}^e = \iint_{A^e} \frac{\partial \mathbf{N}^T}{\partial y} k \frac{\partial \mathbf{N}}{\partial y} dxdy = \iint_{A^e} \begin{bmatrix} m_{31} \\ m_{32} \\ m_{33} \end{bmatrix} k \begin{bmatrix} m_{31} & m_{32} & m_{33} \end{bmatrix} dxdy = kA_e \begin{bmatrix} m_{31}^2 & m_{31} m_{32} & m_{31} m_{33} \\ m_{32} m_{31} & m_{32}^2 & m_{32} m_{33} \\ m_{33} m_{31} & m_{33} m_{32} & m_{33}^2 \end{bmatrix}$$

$$\mathbf{K}_{cvB}^e = \int_{FA} \mathbf{N}^T h_{FA} \mathbf{N}\, dC + \int_{AD} \mathbf{N}^T h_{AD} \mathbf{N}\, dC = \int_{C^e} \begin{bmatrix} L_i \\ L_j \\ L_k \end{bmatrix} h_B \begin{bmatrix} L_i & L_j & L_k \end{bmatrix} dC = \int_{C^e} h_B \begin{bmatrix} L_i^2 & L_i L_j & 0 \\ L_j L_i & L_j^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} dC = \frac{h_B l_{ij}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

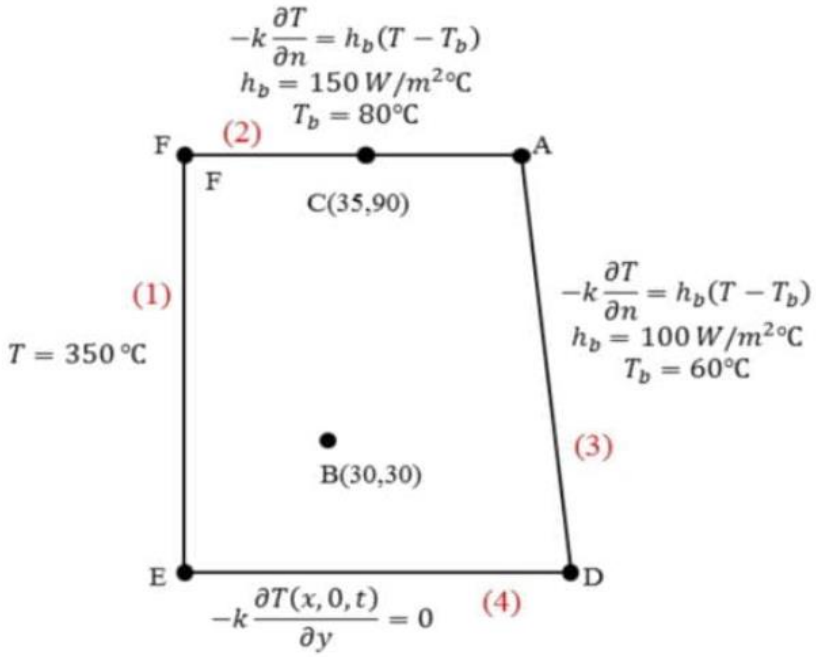
$$\mathbf{f}_{\mathbf{Q}}^e = \iint_{A^e} \mathbf{N}^T Q\, dxdy = \frac{Q A_e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



$$\mathbf{f}_{cvB}^e = \int_F^A \mathbf{N}^T h_{FA} T_{a\,FA}\, dC + \int_A^D \mathbf{N}^T h_{AD} T_{a\,AD}\, dC = \frac{h_B l_{ij} T_{aB}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Problem: Transient Thermal Analysis

$$\bar{T}(x,y,t) = T(x,y,t) - 350 \quad \begin{cases} \bar{T}(x,y,t) = 0 & @\ FE \\ -k\dfrac{\partial \bar{T}}{\partial y} = h_{b2}(\bar{T}(x,y,t) + 350 - 80) = h_{b2}(\bar{T}(x,y,t) + 270) & @\ FA \\ -k\dfrac{\partial \bar{T}}{\partial n} = h_{b3}(\bar{T}(x,y,t) + 350 - 60) = h_{b2}(\bar{T}(x,y,t) + 290) & @\ AD \\ -k\dfrac{\partial \bar{T}}{\partial y} = 0 & @\ DE \end{cases}$$



Problem: Transient Thermal Analysis

$$\bar{T}(x,y,t) = T(x,y,t) - 350 \quad \begin{cases} \bar{T}(x,y,t) = 0 \quad @\, FE \\ -k\dfrac{\partial \bar{T}}{\partial y} = h_{b2}(\bar{T}(x,y,t) + 350 - 80) = h_{b2}(\bar{T}(x,y,t) + 270) \quad @\, FA \\ -k\dfrac{\partial \bar{T}}{\partial n} = h_{b3}(\bar{T}(x,y,t) + 350 - 60) = h_{b2}(\bar{T}(x,y,t) + 290) \quad @\, AD \\ -k\dfrac{\partial \bar{T}}{\partial y} = 0 \quad @\, DE \end{cases}$$

$$[\mathbf{C}]\{\dot{\bar{\mathbf{T}}}(t)\} + [\mathbf{K}]\{\bar{\mathbf{T}}(t)\} = \{\mathbf{F}\} \Rightarrow \begin{bmatrix} [\mathbf{C}_{PP}] & [\mathbf{C}_{PF}] \\ [\mathbf{C}_{FP}] & [\mathbf{C}_{FF}] \end{bmatrix} \begin{Bmatrix} \{\dot{\mathbf{T}}_P(t)\} \\ \{\dot{\mathbf{T}}_F(t)\} \end{Bmatrix} + \begin{bmatrix} [\mathbf{K}_{PP}] & [\mathbf{K}_{PF}] \\ [\mathbf{K}_{FP}] & [\mathbf{K}_{FF}] \end{bmatrix} \begin{Bmatrix} \{\mathbf{T}_P(t)\} \\ \{\mathbf{T}_F(t)\} \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{F}_P\} \\ \{\mathbf{F}_F\} \end{Bmatrix}$$

$$[\mathbf{C}_{PP}]\{\dot{\mathbf{T}}_P(t)\} + [\mathbf{C}_{PF}]\{\dot{\mathbf{T}}_F(t)\} + [\mathbf{K}_{PP}]\{\mathbf{T}_P(t)\} + [\mathbf{K}_{PF}]\{\mathbf{T}_F(t)\} = \{\mathbf{F}_P\}$$

$$[\mathbf{C}_{FP}]\{\dot{\mathbf{T}}_P(t)\} + [\mathbf{C}_{FF}]\{\dot{\mathbf{T}}_F(t)\} + [\mathbf{K}_{FP}]\{\mathbf{T}_P(t)\} + [\mathbf{K}_{FF}]\{\mathbf{T}_F(t)\} = \{\mathbf{F}_F\}$$

$$\Rightarrow \{\mathbf{T}_P(t)\} = 0$$

$$\Downarrow$$

$$[\mathbf{C}_{FF}]\{\dot{\mathbf{T}}_F(t)\} + [\mathbf{K}_{FF}]\{\mathbf{T}_F(t)\} = \{\mathbf{F}_F\}$$

Problem: Transient Thermal Analysis

Data Preparation (Create Input file)

Nodes Coordinates geom(nnd, 2)

Element Connectivity connec(nel, nne)

Material and Geometrical Properties

$$Q = 10^5 \left(\frac{W}{m^3}\right), \quad c = 400 \left(\frac{J}{kg\ C}\right), \quad k = 40 \left(\frac{W}{m\ C}\right)$$

$$\rho = 7800 \left(\frac{kg}{m^3}\right), \quad \mathrm{T}(x, y, t = 0) = 50\ (C)$$

Boundary Conditions nf(nnd, nodof)

Loading

$$\bar{T}_{AD} = -290\ (C), \quad h_{AD} = 100 \left(\frac{W}{m^2\ C}\right)$$

$$\bar{T}_{AF} = -270\ (C), \quad h_{AD} = 150 \left(\frac{W}{m^2\ C}\right)$$

Problem: Transient Thermal Analysis

Apply B.C's and Solve (free) Nodal Displacement

$$[\mathbf{C}]\left\{\dot{\bar{T}}\,(t)\right\} + [\mathbf{K}]\{\bar{\mathbf{T}}(t)\} = \{\mathbf{F}\}$$

$$\begin{bmatrix} [\mathbf{C}_{PP}] & [\mathbf{C}_{\mathbf{PF}}] \\ [\mathbf{C}_{\mathbf{FP}}] & [\mathbf{C}_{\mathbf{FF}}] \end{bmatrix} \begin{Bmatrix} \{\dot{\mathbf{T}}_P(t)\} \\ \{\dot{\mathbf{T}}_F(t)\} \end{Bmatrix} + \begin{bmatrix} [\mathbf{K}_{\mathbf{PP}}] & [\mathbf{K}_{\mathbf{PF}}] \\ [\mathbf{K}_{\mathbf{FP}}] & [\mathbf{K}_{\mathbf{FF}}] \end{bmatrix} \begin{Bmatrix} \{\mathbf{T}_{\mathbf{P}}(t)\} \\ \{\mathbf{T}_{\mathbf{F}}(t)\} \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{F}_{\mathbf{P}}\} \\ \{\mathbf{F}_{\mathbf{F}}\} \end{Bmatrix}$$

$$[\mathbf{C}_{PP}]\{\dot{\mathbf{T}}_P(t)\} + [\mathbf{C}_{\mathbf{PF}}]\{\dot{\mathbf{T}}_F(t)\} + [\mathbf{K}_{\mathbf{PP}}]\{\mathbf{T}_{\mathbf{P}}(t)\} + [\mathbf{K}_{\mathbf{PF}}]\{\mathbf{T}_{\mathbf{F}}(t)\} = \{\mathbf{F}_{\mathbf{P}}\}$$

$$[\mathbf{C}_{\mathbf{FP}}]\{\dot{\mathbf{T}}_P(t)\} + [\mathbf{C}_{\mathbf{FF}}]\{\dot{\mathbf{T}}_F(t)\} + [\mathbf{K}_{\mathbf{FP}}]\{\mathbf{T}_{\mathbf{P}}(t)\} + [\mathbf{K}_{\mathbf{FF}}]\{\mathbf{T}_{\mathbf{F}}(t)\} = \{\mathbf{F}_{\mathbf{F}}\}$$

$\{\mathbf{T}_{\mathbf{P}}(t)\} = 0$

$$[\mathbf{C}_{\mathbf{FF}}]\{\dot{\mathbf{T}}_F(t)\} + [\mathbf{K}_{\mathbf{FF}}]\{\mathbf{T}_{\mathbf{F}}(t)\} = \{\mathbf{F}_{\mathbf{F}}\}$$

MATLAB ODE45

$\{\mathbf{T}_{\mathbf{F}}(t)\}$

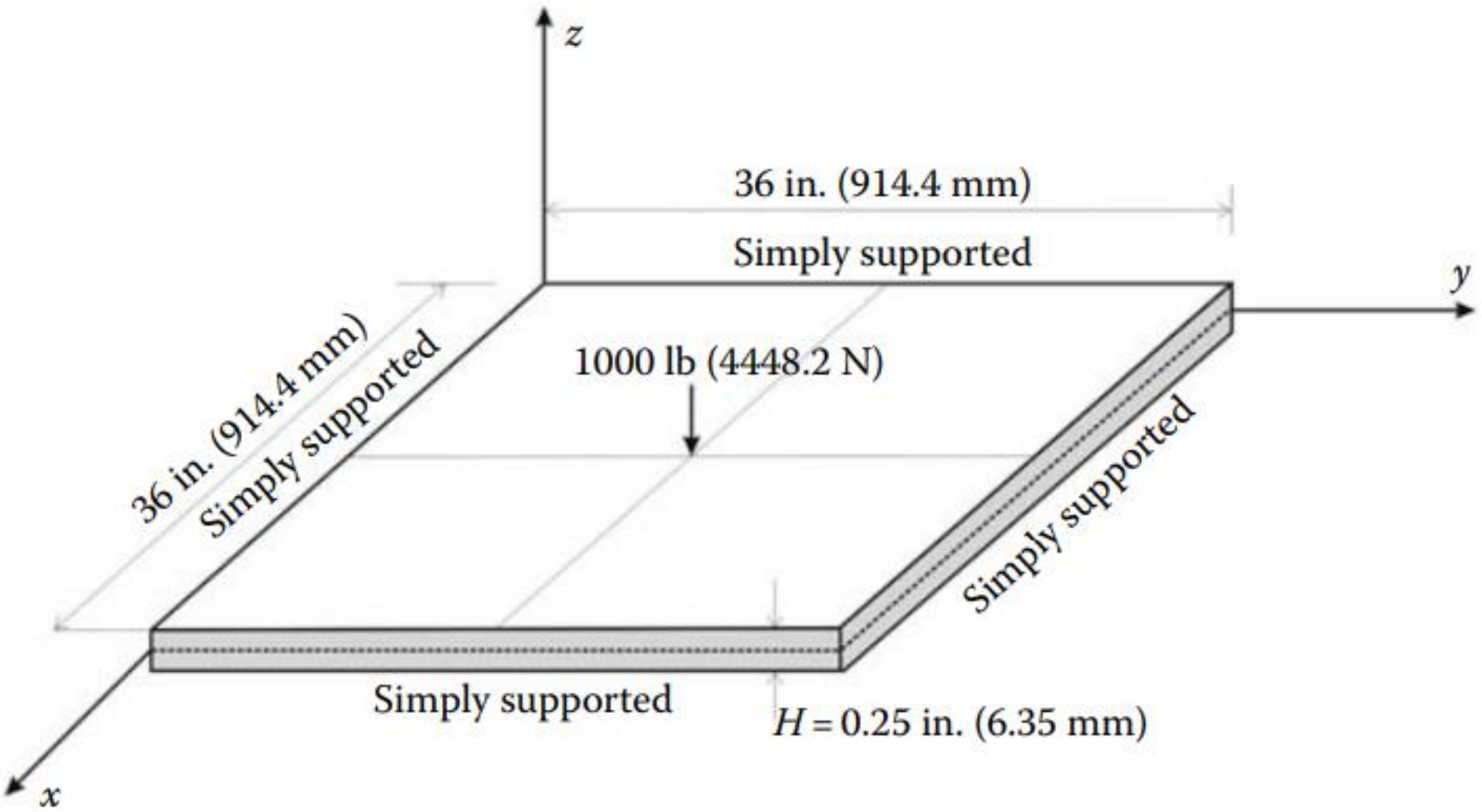
$\dot{x} = f(t, x)$

$\{\dot{T}_F\} = \text{inv(C)*}(\{F_F\} - [K_{FF}]\{T_F\})$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Thin Plate Problem

Problem Description



Plates are structural elements that are bound by two lateral surfaces .The dimensions of the lateral surfaces are very large compared to the thickness of the plate. A plate may be thought of as the two-dimensional equivalent of a beam. Plates are also generally subject to loads normal to their plane.

Thin Plate Problem

The small deflection theory of plates attributed to Kirchhoff is based on the following assumptions:

1. The x–y plane coincides with the middle plane of the plate in the undeformed geometry.

2. The lateral dimension of the plate is at least **10** times its thickness.

3. The vertical displacement of any point of the plate can be taken equal to that of the point (below or above it) in the middle plane.

4. A vertical element of the plate before bending remains perpendicular to the middle surface of the plate after bending. ⇨ $\gamma_{xz} = \gamma_{yz} = 0$

5. **Strains are small**: deflections are less than the order of (1/100) of the span length.

6. The strain of the middle surface is zero or negligible.

Thin Plate Problem

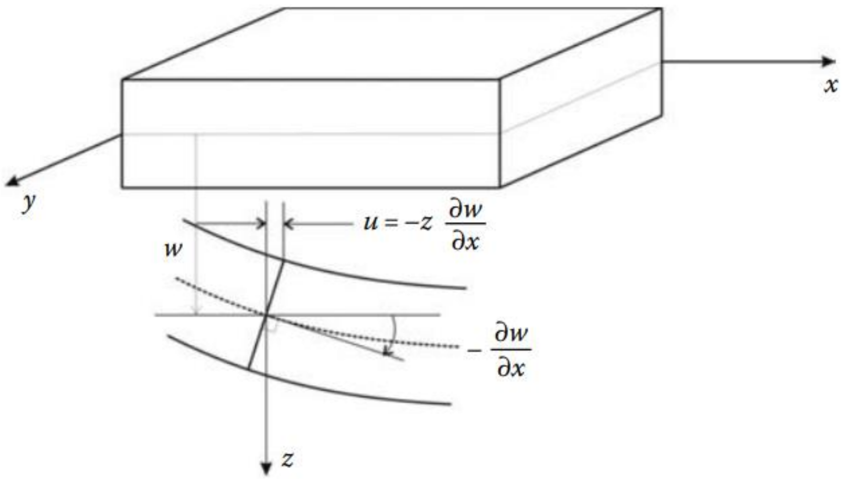
Considering the plate element shown in Figure, the in-plane displacements u and v, respectively in the directions x and y, can be expressed as

$$\gamma_{xz} = 0 \qquad \Rightarrow \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad \Rightarrow \qquad u = -z\frac{\partial w}{\partial x}$$

$$\gamma_{yz} = 0 \qquad \Rightarrow \qquad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \qquad \Rightarrow \qquad v = -z\frac{\partial w}{\partial y}$$

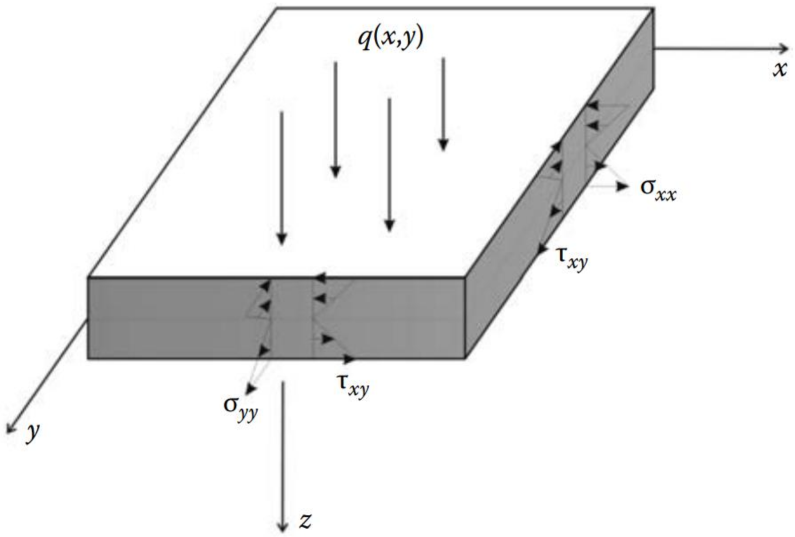
$$\Rightarrow \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} -z\frac{\partial^2 w}{\partial x^2} \\ -z\frac{\partial^2 w}{\partial y^2} \\ -2z\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = -z\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix}$$

The vector $\{\chi\} = \left[\chi_x \; \chi_y \; \chi_{xy}\right]^T$ is called the vector of curvature or **generalized strain**



Thin Plate Problem

Internal stresses in a thin plate. Moments and shear forces due to internal stresses in a thin plate.

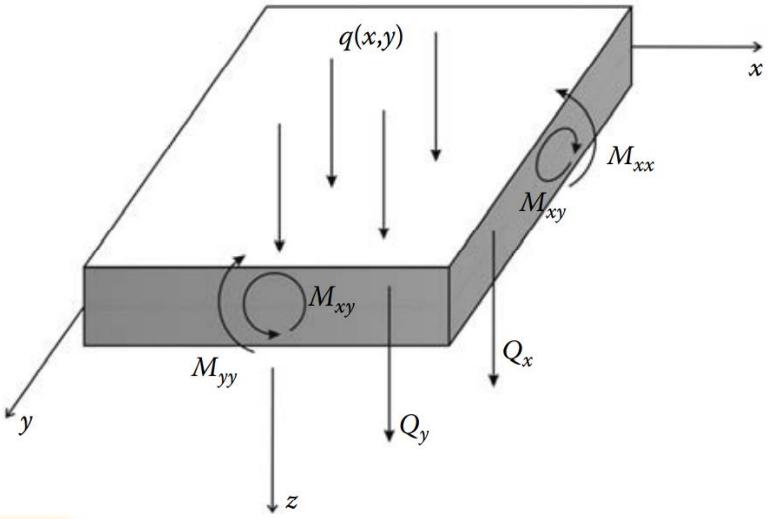


Moments and shear forces due to internal stresses in a thin plate.

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z\, dz \qquad Q_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} dz$$

$$M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z\, dz \qquad Q_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz$$



Thin Plate Problem

Internal stresses in plates produce bending moments and shear forces as illustrated in Figures. The moments and shear forces are the resultants of the stresses and are defined as acting per unit length of plate. These internal actions are defined as

Assuming a state of plane stress conditions for plate bending

$$\{\sigma\} = [D]\{\epsilon\}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$\{\sigma\} = -z[D]\{\chi\}$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z \, dz$$

$$M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z \, dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz$$

$$Q_{xx} = \int_{-h/2}^{h/2} \sigma_{xz} dz$$

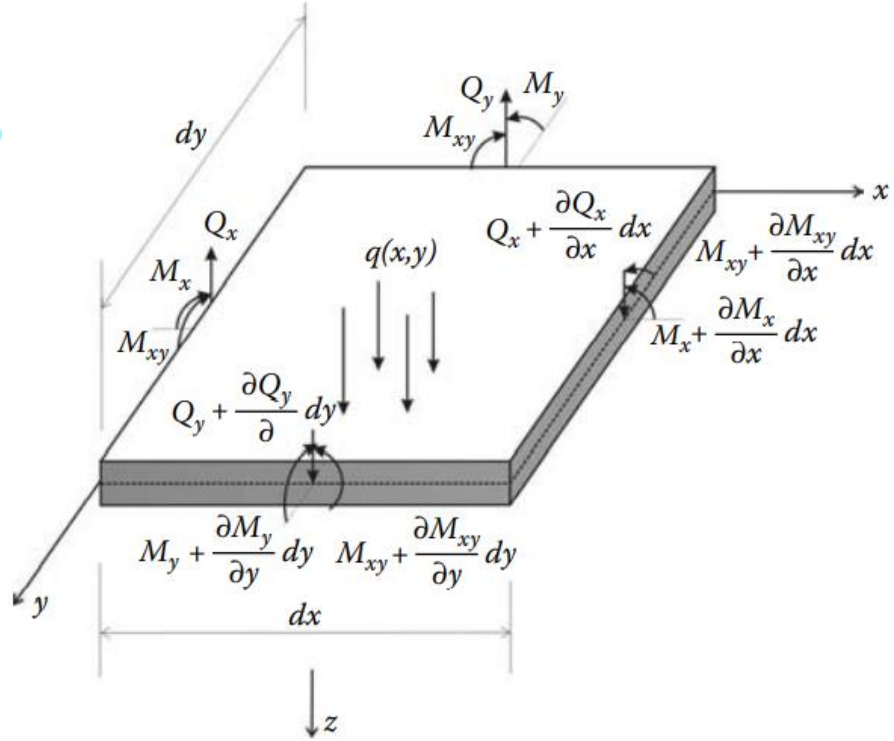
$$Q_{yy} = \int_{-h/2}^{h/2} \sigma_{yz} dz$$

$$\{M\} = \frac{h^3}{12}[D]\{\chi\}$$

Thin Plate Problem

Consider the equilibrium of the free body of the differential plate element shown in Figure Recalling that Q_x represents force per unit length along the edge dy and requiring force equilibrium in z direction results in

$$-Q_x dy - Q_y dx + \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx + q(x,y)dxdy = 0$$



Moment equilibrium about the y-axis leads to

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_{xx}}{\partial x} = Q_x$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x,y) = 0$$

GOVERNING EQUATION IN TERMS OF DISPLACEMENT VARIABLES

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial^2 x \partial^2 y} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D_r} \quad \Rightarrow \quad \nabla^4 w = \frac{q}{D_r}$$

$$D_r = \frac{Eh^3}{12(1-\nu^2)}$$

Thin Plate Problem

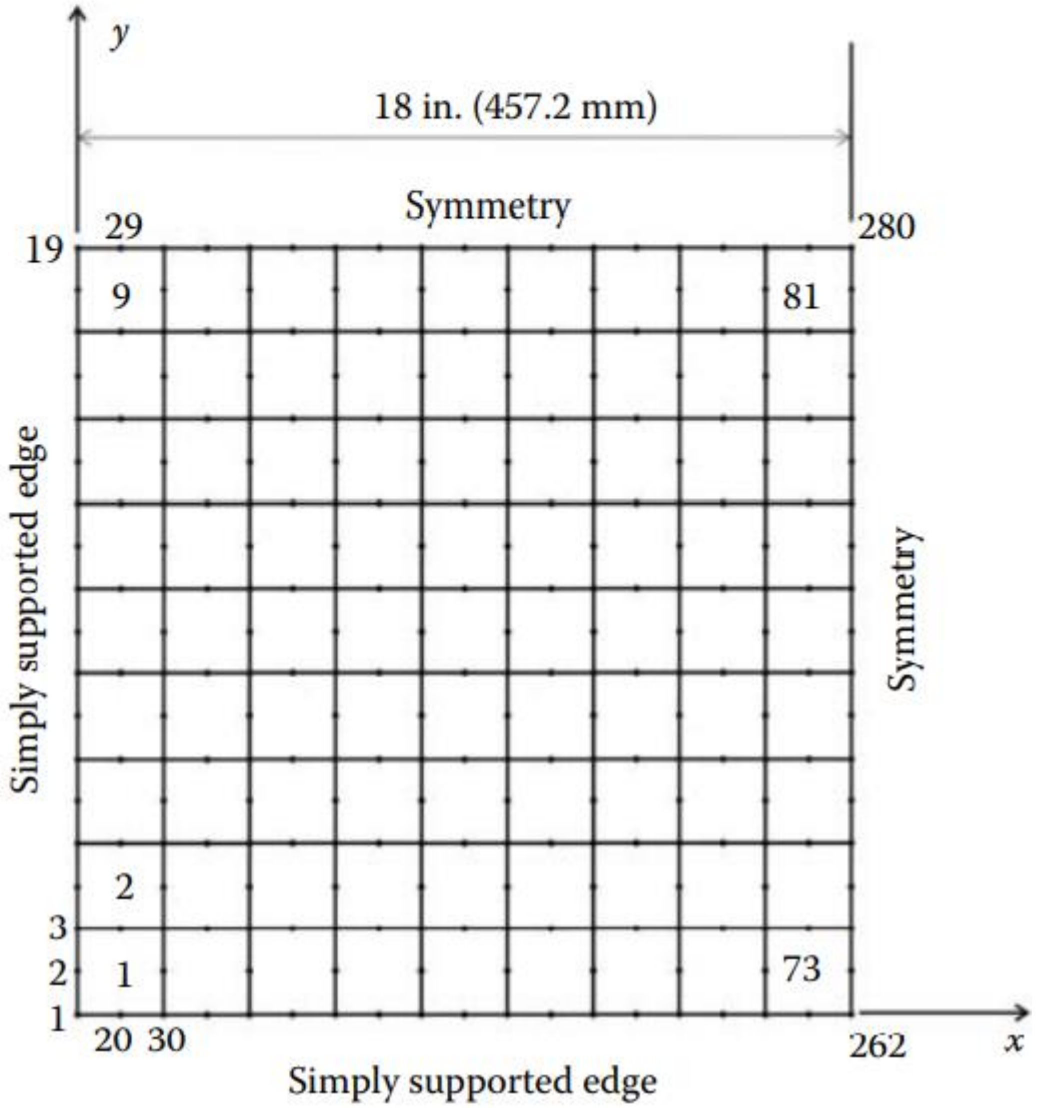
$$\{M\} = \frac{h^3}{12}[D]\{\chi\}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$U = \frac{1}{2}\{\chi\}^T[D]\{\chi\}\,dA$$

Thin Plate Problem

Discretization: Mesh Generation



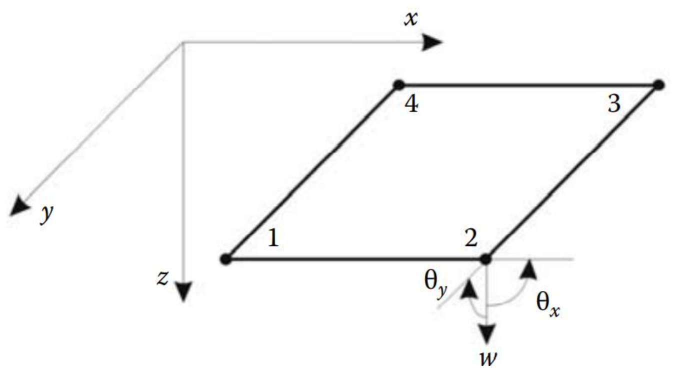
Thin Plate Problem

Rectangular Element: Interpolation

The element has four nodes and 12 DOF in total → A trial function will contain 12 parameters

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3$$

$$w(x_1,y_1) = w_1$$
$$w(x_2,y_2) = w_2$$
$$w(x_3,y_3) = w_3$$
$$w(x_4,y_4) = w_4$$



$$\theta_x(x,y) = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_4 x + \alpha_5 y + 3\alpha_7 x^2 + 2\alpha_8 xy + \alpha_9 y^2 + 3\alpha_{11} x^2 y + \alpha_{12} y$$

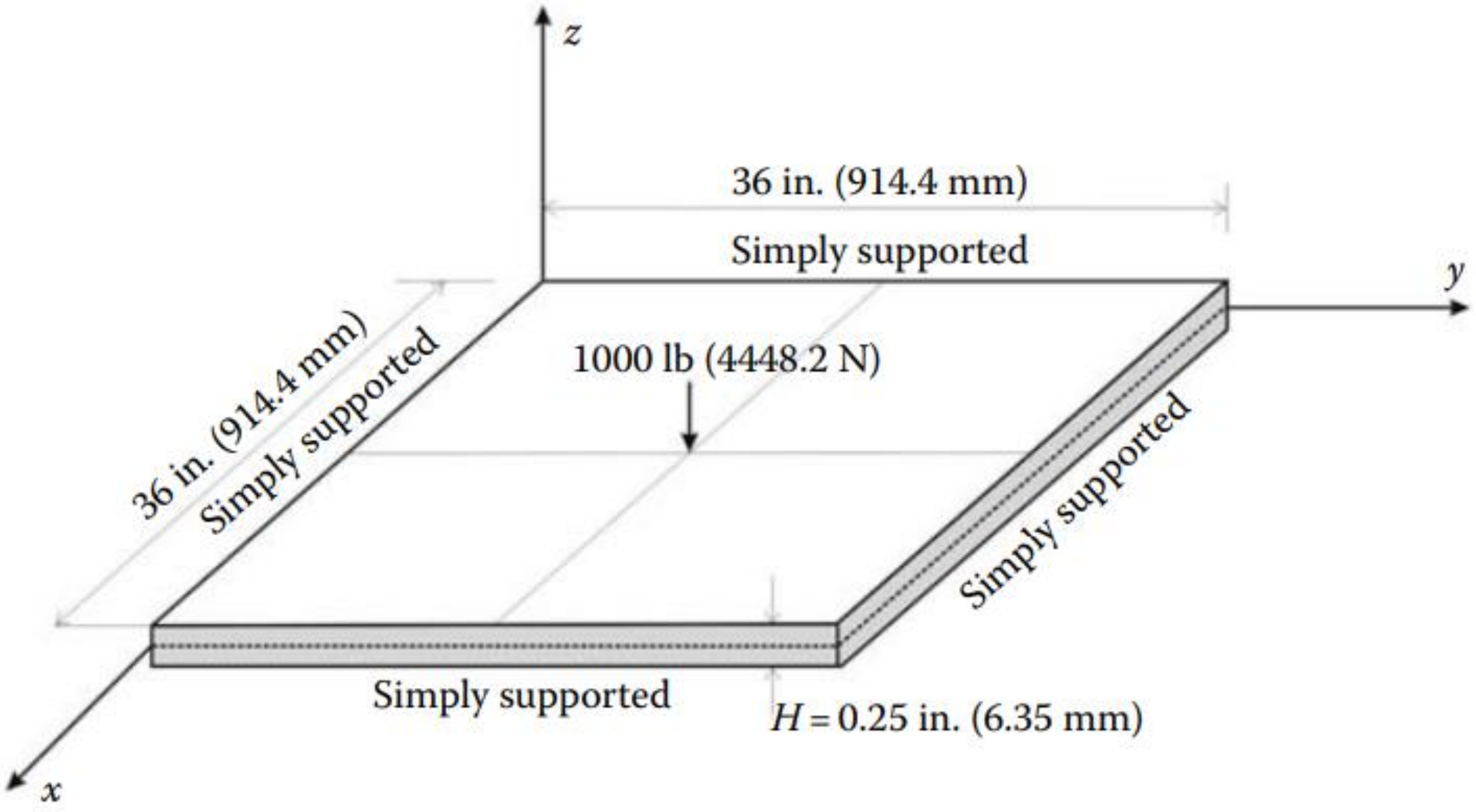
$$\theta_x(x_1,y_1) = \theta_{x1} \qquad \theta_x(x_3,y_3) = \theta_{x3}$$
$$\theta_x(x_2,y_2) = \theta_{x2} \qquad \theta_x(x_4,y_4) = \theta_{x4}$$

$$\theta_y(x,y) = \frac{\partial w}{\partial y} = \alpha_3 + \alpha_5 x + 2\alpha_6 y + \alpha_8 x^2 + 2\alpha_9 xy + 3\alpha_{10} y^2 + \alpha_{11} x^3 + 3\alpha_{12} xy^2$$

$$\theta_y(x_1,y_1) = \theta_{y1} \qquad \theta_y(x_3,y_3) = \theta_{y3}$$
$$\theta_y(x_2,y_2) = \theta_{y2} \qquad \theta_y(x_4,y_4) = \theta_{y4}$$

Thick Plate Problem
(Mindlin Plate Theory)

Problem Description



Thick Plate Problem

Consistent units

Quantity	SI	SI (mm)	US Unit (ft)	US Unit (inch)
Length	m	mm	ft	in
Force	N	N	lbf	lbf
Mass	kg	tonne (10^3 kg)	slug	lbf s^2/in
Time	s	s	s	s
Stress	Pa (N/m^2)	MPa (N/mm^2)	lbf/ft^2	psi (lbf/in^2)
Energy	J	mJ (10^{-3} J)	ft lbf	in lbf
Density	kg/m^3	$tonne/mm^3$	$slug/ft^3$	lbf s^2/in^4

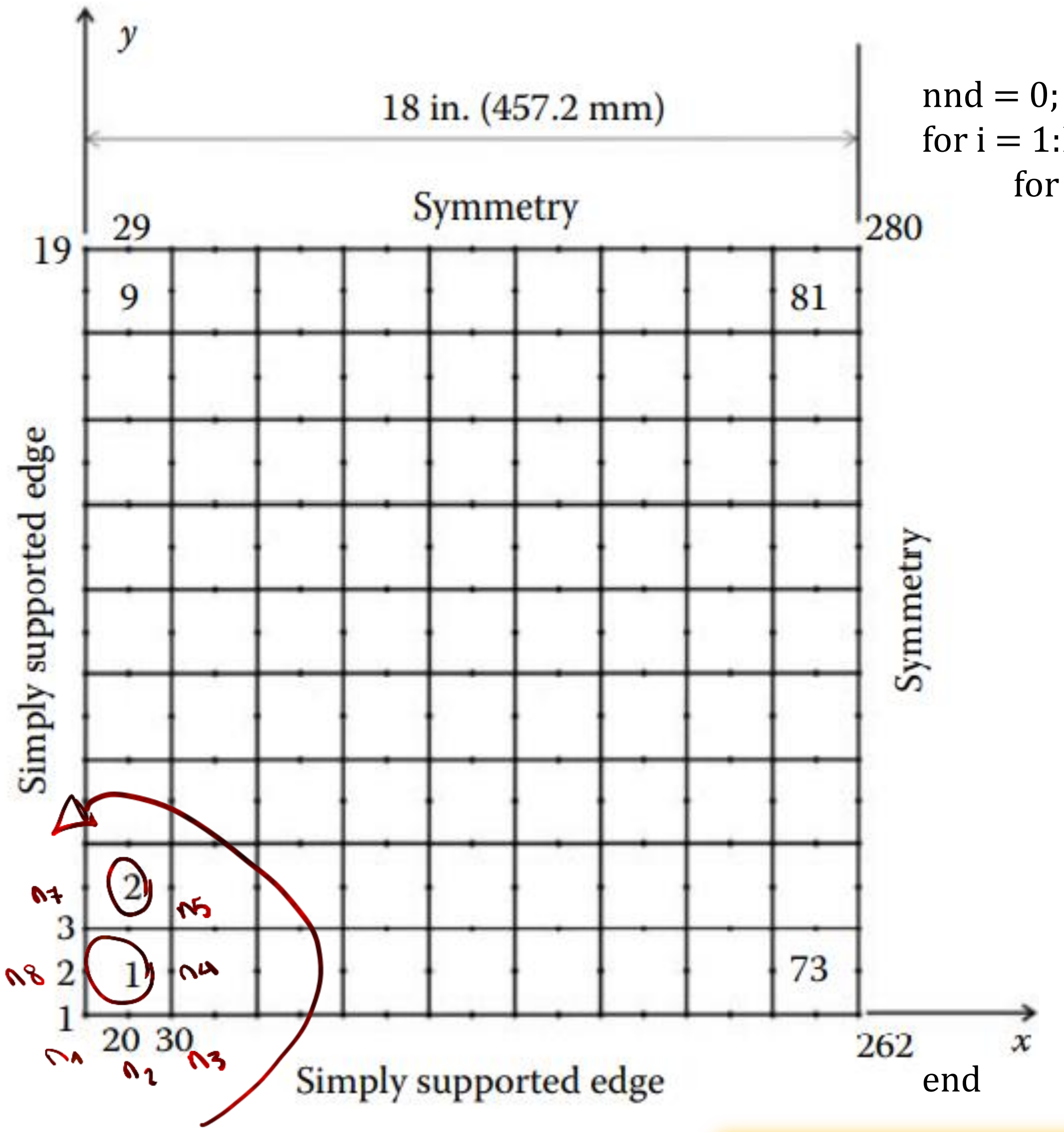
Thick Plate Problem

Data Preparation (Create Input file)

Nodes Coordinates	geom(nnd, 2)
Element Connectivity	connec(nel, nne)
Material and Geometrical Properties	$E = 30 \times 10^6\ (psi) \quad \nu = 0.3$
Boundary Conditions	nf(nnd, nodof)
Loading	The force in the global force vector **F_f**

Thick Plate Problem

Discretization: Mesh Generation



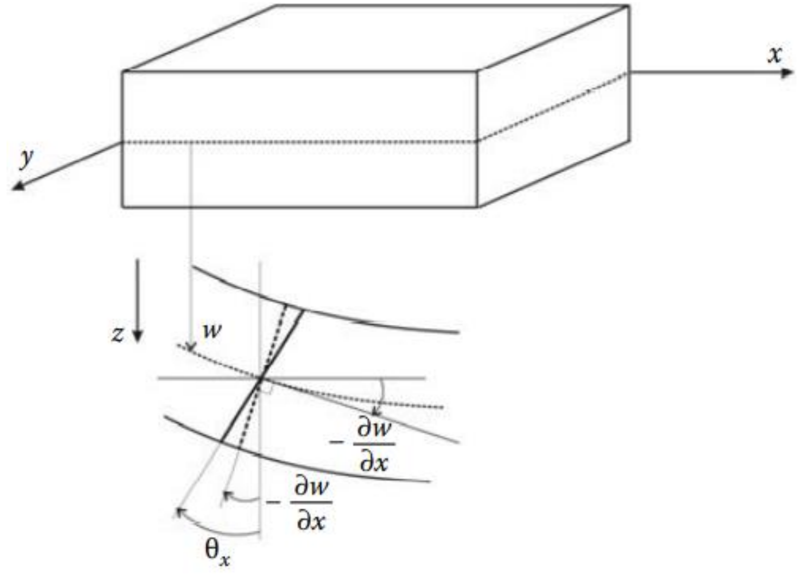

```
nnd = 0;     k = 0;
for i = 1:NXE
        for j=1:NYE
                k = k + 1;
                %
                n1 = (i-1)*(3*NYE+2)+2*j - 1;      n8 = n1 + 1;          n7 = n1 + 2;
                n2 = i*(3*NYE+2)+j - NYE - 1;      n6 = n2 + 1;
                n3 = i*(3*NYE+2)+2*j-1;            n4 = n3 + 1;          n5 = n3 + 2;
                %
                geom(n1,:) = [(i-1)*dhx - X_origin (j-1)*dhy - Y_origin ];
                geom(n3,:) = [i*dhx - X_origin (j-1)*dhy - Y_origin ];
                geom(n2,:) = [(geom(n1,1)+geom(n3,1))/2 (geom(n1,2)+geom(n3,2))/2];
                geom(n5,:) = [i*dhx- X_origin j*dhy - Y_origin ];
                geom(n4,:) = [(geom(n3,1)+ geom(n5,1))/2 (geom(n3,2)+ geom(n5,2))/2];
                geom(n7,:) = [(i-1)*dhx - X_origin j*dhy - Y_origin ];
                geom(n6,:) = [(geom(n5,1)+ geom(n7,1))/2 (geom(n5,2)+ geom(n7,2))/2];
                geom(n8,:) = [(geom(n1,1)+ geom(n7,1))/2 (geom(n1,2)+ geom(n7,2))/2];
                %
                nel = k;
                nnd = n5;
                connec(k,:) = [n1 n2 n3 n4 n5 n6 n7 n8];
                end
end
```

Thick Plate Problem

In thick plates, the assumption that a vertical element of the plate before bending remains perpendicular to the middle surface of the plate after bending is relaxed. Transverse normal may rotate without remaining normal to the mid-plane. A line originally normal to the middle plane will develop rotation components θ_x relative to the middle plane after deformation as shown in Figure. A similar definition holds for θ_y. Hence, the displacement field becomes

$$u = z\theta_x$$
$$v = z\theta_y$$
$$w = w(x, y)$$

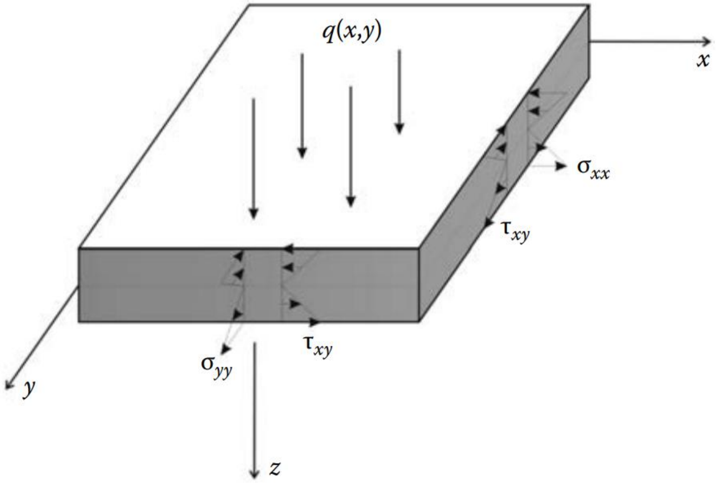
$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} z\dfrac{\partial\theta_x}{\partial x} \\ z\dfrac{\partial\theta_y}{\partial y} \\ z\left(\dfrac{\partial\theta_x}{\partial y} + \dfrac{\partial\theta_y}{\partial x}\right) \\ z\left(\theta_y - \dfrac{\partial w}{\partial y}\right) \\ z\left(\theta_x - \dfrac{\partial w}{\partial x}\right) \end{Bmatrix}$$



These equations are the main equations of the Mindlin plate theory. The theory accounts for transverse shear deformations and is applicable for moderately thick plates. Unlike in thin plate theory, it is important to notice that the transverse displacement $w(x, y)$ and slopes θ_x, θ_y are independent. Notice also that the thick plate theory reduces to thin plate theory if $\theta_x = -\partial w/\partial x$ and $\theta_y = -\partial w/\partial y$.

Thick Plate Problem

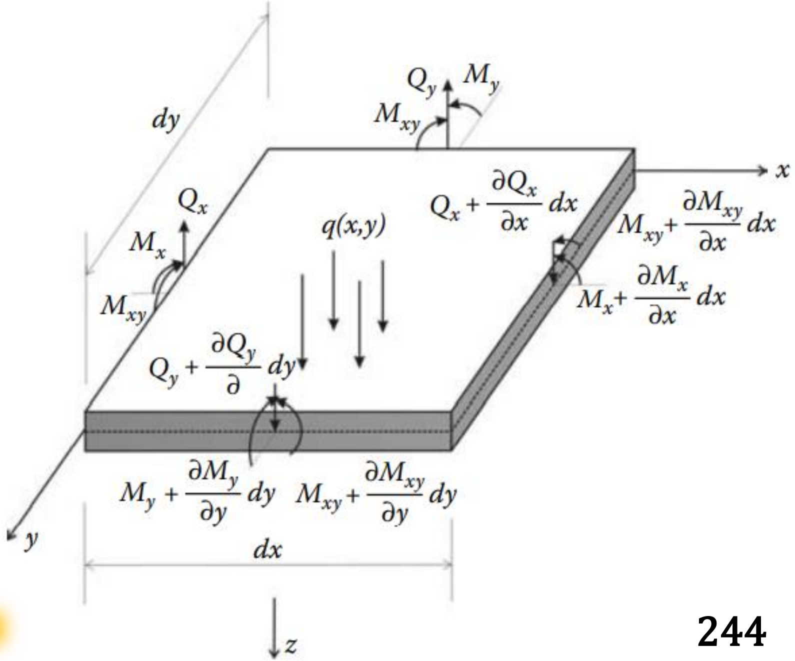
Consider the equilibrium of the free body of the differential plate element shown in Figure Recalling that Q_x represents force per unit length along the edge dy and requiring force equilibrium in z direction results in



$$-Q_x dy - Q_y dx + \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx + q(x,y)dxdy = 0$$

Moment equilibrium about the y-axis leads to

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_{xx}}{\partial x} = Q_x$$



$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x,y) = 0$$

Thick Plate Problem

STRESS–STRAIN RELATIONSHIP

Assuming the material is homogeneous and isotropic, the plane stresses σ_{xx}, σ_{yy}, and τ_{xy} are related to the strains through the elasticity matrix [D].The shear strains τ_{yz} and τ_{xz} are related to the shear strains γ_{yz} and γ_{xz} through

$$\{\sigma\} = [D]\{\epsilon\}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

$$G = \frac{E}{2(1+\nu)}$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z \, dz$$

$$M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z \, dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz$$

$$Q_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} dz$$

$$Q_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} D_r & \nu \times D_r & 0 & 0 & 0 \\ \nu \times D_r & D_r & 0 & 0 & 0 \\ 0 & 0 & \frac{D_r(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & Gh & 0 \\ 0 & 0 & 0 & 0 & Gh \end{bmatrix} \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \\ \left(\theta_y - \frac{\partial w}{\partial y}\right) \\ \left(\theta_x - \frac{\partial w}{\partial x}\right) \end{Bmatrix}$$

$$D_r = \frac{Eh^3}{12(1-\nu^2)}$$

χ

$\begin{Bmatrix} M \\ Q \end{Bmatrix} = [\quad]\{\chi\}$

Thick Plate Problem

The Equation can be written more compactly as

$$\{M\} = [D_M]\{\chi\}$$

The total strain energy of the plate is given as

$$U = \frac{1}{2}\int_A \{\chi\}^T[D_M]\{\chi\}\, dA \quad \Rightarrow \quad U = U_B + U_S = \frac{1}{2}\int_A \{\chi_B\}^T[D_B]\{\chi_B\}\, dA + \frac{\kappa}{2}\int_A \{\chi_S\}^T[D_S]\{\chi_S\}\, dA$$

$K = \int B^T D B$

$U = 1/2\, q\, K_{eq}\, q \sim K$

κ is the shear energy correction factor equal to 5/6

$$\{\chi_B\} = \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \left(\dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x}\right) \end{Bmatrix} \qquad \{\chi_S\} = \begin{Bmatrix} \left(\theta_y - \dfrac{\partial w}{\partial y}\right) \\ \left(\theta_x - \dfrac{\partial w}{\partial x}\right) \end{Bmatrix}$$

$$[D_B] = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \qquad [D_S] = G\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$$

Thick Plate Problem

Rectangular Element: Interpolation

The element has 8 nodes and 24 DOF in total → A trial function will contain 24 parameters

C^0 iso-parametric shape functions can be used for the thick plate element formulation

Eight-nodded Iso-parametric Element

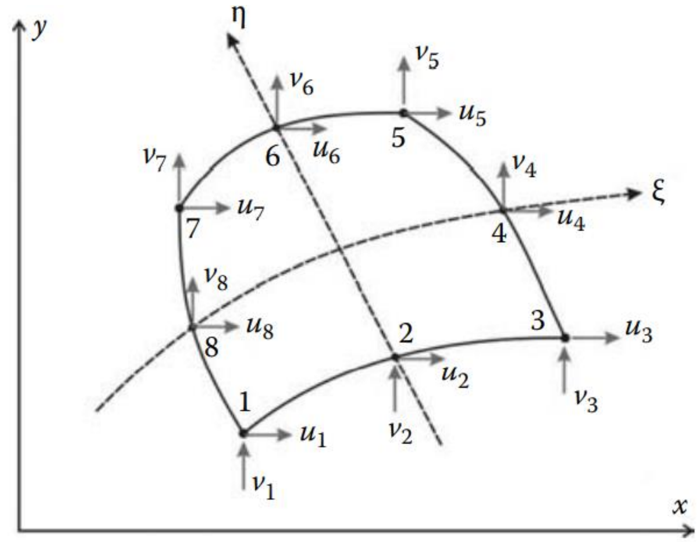
$w(1,-1) = \quad = w_1$

$$w = \sum_{i=1}^{n} N_i(\xi,\eta) w_i$$

$$\theta_x = \sum_{i=1}^{n} N_i(\xi,\eta) \theta_{xi}$$

$$\theta_y = \sum_{i=1}^{n} N_i(\xi,\eta) \theta_{yi}$$

$$\begin{Bmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ N_3(\xi,\eta) \\ N_4(\xi,\eta) \\ N_5(\xi,\eta) \\ N_6(\xi,\eta) \\ N_7(\xi,\eta) \\ N_8(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} -0.25(1-\xi)(1-\eta)(1+\xi+\eta) \\ 0.50(1-\xi^2)(1-\eta) \\ -0.25(1+\xi)(1-\eta)(1-\xi+\eta) \\ 0.50(1+\xi)(1-\eta^2) \\ -0.25(1+\xi)(1+\eta)(1-\xi-\eta) \\ 0.50(1-\xi^2)(1+\eta) \\ -0.25(1-\xi)(1+\eta)(1+\xi-\eta) \\ 0.50(1-\xi)(1-\eta^2) \end{Bmatrix}$$



$$w(x,y) = N_1(\xi,\eta)w_1 + N_3(\xi,\eta)w_2 + N_3(\xi,\eta)w_3 + N_4(\xi,\eta)w_4 + N_5(\xi,\eta)w_5 + N_6(\xi,\eta)w_6 + N_7(\xi,\eta)w_7 + N_8(\xi,\eta)w_8$$
$$\theta_x(x,y) = N_1(\xi,\eta)\theta_{x1} + N_3(\xi,\eta)\theta_{x2} + N_3(\xi,\eta)\theta_{x3} + N_4(\xi,\eta)\theta_{x4} + N_5(\xi,\eta)\theta_{x5} + N_6(\xi,\eta)\theta_{x6} + N_7(\xi,\eta)\theta_{x7} + N_8(\xi,\eta)\theta_{x8}$$
$$\theta_y(x,y) = N_1(\xi,\eta)\theta_{y1} + N_3(\xi,\eta)\theta_{y2} + N_3(\xi,\eta)\theta_{y3} + N_4(\xi,\eta)\theta_{y4} + N_5(\xi,\eta)\theta_{y5} + N_6(\xi,\eta)\theta_{y6} + N_7(\xi,\eta)\theta_{y7} + N_8(\xi,\eta)\theta_{y8}$$

Thick Plate Problem

Strain Energy: $U = U_B + U_S = \frac{1}{2}\int_A \{\chi_B\}^T[D_B]\{\chi_B\}\,dA + \frac{\kappa}{2}\int_A \{\chi_S\}^T[D_S]\{\chi_S\}\,dA$ $\quad (\kappa = 5/6)$

$$\{\chi\}_B = [L_B][N]\{a\} = [B_B]\{a\} \qquad \{\chi\}_S = [L_S][N]\{a\} = [B_S]\{a\}$$

$$\{a\} = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad | \quad \dots \quad \dots \quad \dots \quad | \quad w_n \quad \theta_{xn} \quad \theta_{yn}]^T$$

$\frac{1}{2}\int ([B_B]\{a\})^T [D_B]([B_B]\{a\})\,dA$

$K = \int B^T D B\,dA$

$$[L_B] = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad [N] = \begin{bmatrix} N_1 & 0 & 0 & | & \dots & \dots & \dots & | & N_n & 0 & 0 \\ 0 & N_1 & 0 & | & \dots & \dots & \dots & | & 0 & N_n & 0 \\ 0 & 0 & N_1 & | & \dots & \dots & \dots & | & 0 & 0 & N_n \end{bmatrix} \quad [L_S] = \begin{bmatrix} -\frac{\partial}{\partial y} & 0 & 1 \\ -\frac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

$$[B_B] = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial x} & 0 & | & \dots & \dots & \dots & | & 0 & \frac{\partial N_n}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial y} & | & \dots & \dots & \dots & | & 0 & 0 & \frac{\partial N_n}{\partial y} \\ 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & | & \dots & \dots & \dots & | & 0 & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

$$[B_S] = \begin{bmatrix} -\frac{\partial N_1}{\partial y} & 0 & N_1 & | & \dots & \dots & \dots & | & -\frac{\partial N_n}{\partial y} & 0 & N_n \\ -\frac{\partial N_1}{\partial x} & N_1 & 0 & | & \dots & \dots & \dots & | & -\frac{\partial N_n}{\partial x} & N_n & 0 \end{bmatrix}$$

$$[K_e] = [K_B] + [K_S] = \int_{A_e} [B_B]^T[D_B][B_B]\,dA + \kappa \int_{A_e} [B_S]^T[D_S][B_S]\,dA \qquad (\kappa = 5/6)$$

Thick Plate Problem

Remark: It is important to note that the shear stiffness $[K_S]$ is a function of h since $[D_S]$ is a function of h, and the bending stiffness $[K_B]$ is a function of h^3 since $[D_B]$ is a function of h^3 . A consequence of this is that the **shear energy dominates as the thickness of the plate becomes very small compared to its side length**. This is called shear locking. One way of resolving this problem is to under integrate the shear energy term. For example, if the 8 node quadrilateral is used, then the bending energy is to be integrated with 3 × 3 Gauss points, while the shear energy is to be integrated only with a 2 × 2 rule.

$x(\xi,\eta)$

$$\frac{\partial N}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial N}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial N}{\partial y}$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{8} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

der coord

$$[J] = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \dfrac{\partial N_2}{\partial \xi} & \cdots & \dfrac{\partial N_8}{\partial \xi} \\ \dfrac{\partial N_1}{\partial \eta} & \dfrac{\partial N_2}{\partial \eta} & \cdots & \dfrac{\partial N_8}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_8 & y_8 \end{bmatrix}$$

$$\begin{cases} x = N_1 x_1 + N_2 x_2 + \cdots + N_8 x_8 \\ y = N_1 y_1 + N_2 y_2 + \cdots + N_8 y_8 \end{cases}$$

Der ←

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

Thick Plate Problem

Stiffness Matrix

$$u = z\theta_x$$
$$v = z\theta_y$$
$$w = w(x, y)$$

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} z\dfrac{\partial \theta_x}{\partial x} \\ z\dfrac{\partial \theta_y}{\partial y} \\ z\left(\dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x}\right) \\ z\left(\theta_y - \dfrac{\partial w}{\partial y}\right) \\ z\left(\theta_x - \dfrac{\partial w}{\partial x}\right) \end{Bmatrix}$$

$$w = \sum_{i=1}^{n} N_i(\xi, \eta) w_i$$
$$\theta_x = \sum_{i=1}^{n} N_i(\xi, \eta) \theta_{xi}$$
$$\theta_y = \sum_{i=1}^{n} N_i(\xi, \eta) \theta_{yi}$$

$$[B_B] = \begin{bmatrix} 0 & \dfrac{\partial N_1}{\partial x} & 0 & | & \dots & \dots & \dots & | & 0 & \dfrac{\partial N_n}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial N_1}{\partial y} & | & \dots & \dots & \dots & | & 0 & 0 & \dfrac{\partial N_n}{\partial y} \\ 0 & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & | & \dots & \dots & \dots & | & 0 & \dfrac{\partial N_n}{\partial y} & \dfrac{\partial N_n}{\partial x} \end{bmatrix}$$

$$[B_S] = \begin{bmatrix} -\dfrac{\partial N_1}{\partial y} & 0 & N_1 & | & \dots & \dots & \dots & | & -\dfrac{\partial N_n}{\partial y} & 0 & N_n \\ -\dfrac{\partial N_1}{\partial x} & N_1 & 0 & | & \dots & \dots & \dots & | & -\dfrac{\partial N_n}{\partial x} & N_n & 0 \end{bmatrix}$$

Thick Plate Problem

Stiffness Matrix

$$[K_e]\{a\} = f_e$$

$$[K_e] = \left[\int_{A_e} [B]^T[D][B]\ dA\right]$$

$$\{f_e\} = \int_{A_e} [N]^T\{b\}\ dA + \int_{L_e} [N]^T\{t\}\ dl + \sum_i [N_{(\{x\}=\{\bar{x}\})}]^T\{P\}_i$$

Next Slide

$$[K_e] = \int_{-1}^{+1}\int_{-1}^{+1} [B(\xi,\eta)]^T[D][B(\xi,\eta)]det[J(\xi,\eta)]d\eta\, d\xi$$

$$= \sum_{i=1}^{ngp}\sum_{j=1}^{ngp} W_iW_j[B(\xi_i,\eta_j)]^T[D][B(\xi_i,\eta_j)]det[J(\xi_i,\eta_j)]$$

Thick Plate Problem

Force vector

Body Forces

$$\int_{A_e} [N]^T \{b\}\ dA = \sum_{i=1}^{ngp}\sum_{j=1}^{ngp} W_i W_j [N(\xi_i,\eta_j)]^T \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} det[J(\xi_i,\eta_j)]$$

Traction Forces

$$q_x = q_t dL\cos\alpha - q_n dL\sin\alpha = q_t dx - q_n dy$$

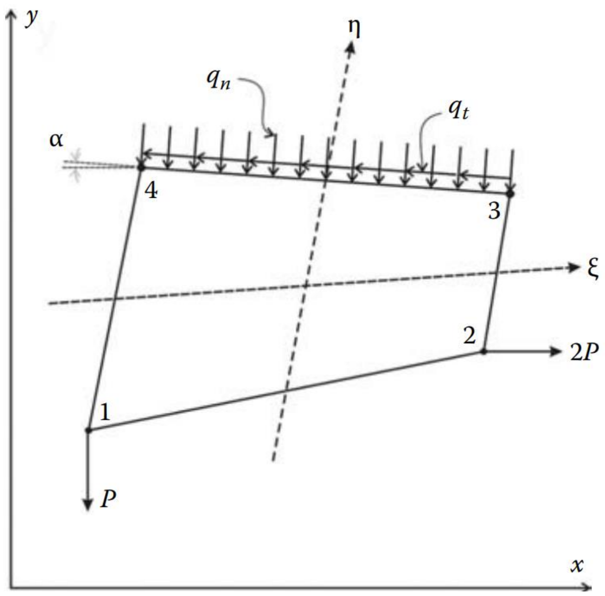
$$q_y = q_n dL\cos\alpha + q_t dL\sin\alpha = q_n dx + q_t dy$$

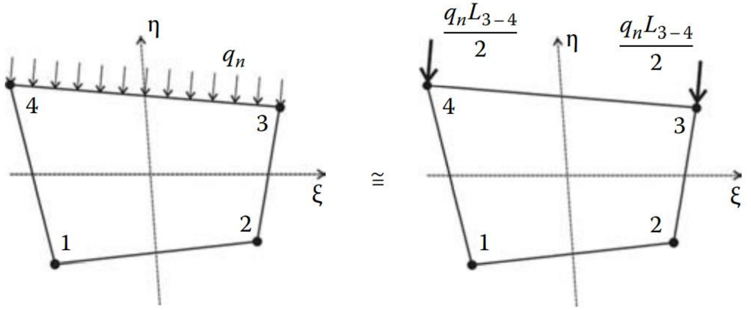
$$q_x = \left(q_t\frac{\partial x}{\partial\xi} - q_n\frac{\partial y}{\partial\xi}\right)d\xi$$

$$q_y = \left(q_n\frac{\partial x}{\partial\xi} + q_t\frac{\partial y}{\partial\xi}\right)d\xi$$

$$\int_{A_e} [N]^T \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} dA = \int_{L_{3-4}} [N(\xi,+1)]^T \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} dl$$

$$= \sum_{i=1}^{ngp} W_i [N(\xi_i,+1)]^T \begin{Bmatrix} \left(q_t\frac{\partial x(\xi_i,+1)}{\partial\xi} - q_n\frac{\partial y(\xi_i,+1)}{\partial\xi}\right) \\ \left(q_n\frac{\partial x(\xi_i,+1)}{\partial\xi} + q_t\frac{\partial y(\xi_i,+1)}{\partial\xi}\right) \end{Bmatrix}$$





Concentrated Forces

$$\sum_{k=1} [N]_{x=x_k}\{P_k\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -P \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 2P \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 2P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thick Plate Problem

Apply B.C's and Solve (free) Nodal Displacement

$$\begin{bmatrix} [K_{PP}] & \vdots & [K_{PF}] \\ \cdots & \cdots & \cdots \\ [K_{FP}] & \vdots & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \{\delta_P\} \\ \cdots \\ \{\delta_F\} \end{Bmatrix} = \begin{Bmatrix} \{F_P\} \\ \cdots \\ \{F_F\} \end{Bmatrix}$$

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

$$[K_{FP}]\{\delta_P\} + [K_{FF}]\{\delta_F\} = \{F_F\}$$

0

linsolve

$$\{\delta_F\} = [K_{FF}]^{-1}\{\{F_F\} - [K_{FP}]\{\delta_P\}\}$$

If $\{\delta_p\} = 0$

$$\{\delta_F\} = [K_{FF}]^{-1}\{F_F\}$$

The subscripts P and F refer respectively to the prescribed and free degrees of freedom

Thick Plate Problem

Calculation of the Element Resultants

SUPPORT REACTIONS

$$[K_{PP}]\{\delta_P\} + [K_{PF}]\{\delta_F\} = \{F_P\}$$

If $\{\delta_p\} = 0$ →

$$\{F_P\} = [K_{PF}]\{\delta_F\}$$


Thanks for attention
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